MA 266 - Exam # 2

Name SOLUTIONS PUID# ____________

(10 pts) **1.** Find the general solution of these linear homogeneous equations with constant coefficients:

(a) $y^{(4)}+2y'''-3y''=0$

$r^4+2r^3-3r^2=0$

$r^2(r^2+2r-3)=0$

$r^2(r+3)(r-1)=0$

$y = \boxed{C_1+C_2t+C_3e^{-3t}+C_4e^t}$

(b) $y''+9y=0$

$r^2+9=0$

$(r-3i)(r+3i)=0$

$y = \boxed{C_1\cos 3t + C_2 \sin 3t}$

(5 pts) **2.** Given that the differential equation $y^{(5)}+y''''+2y'''-22y''-35y'-75y=0 \quad (*)$ has characteristic equation $(\boldsymbol{r^2+2r+5)^2(r-3)=0}$, the general solution of $(*)$ is

$r^2+2r+5=0$

$r=\dfrac{-2\pm\sqrt{4-20}}{2}$

$r=-1+2i$

repeated 2 times

$\Rightarrow e^{-t}\cos t,\ te^{-t}\cos t$

$e^{-t}\sin t,\ te^{-t}\sin t$

A. $y=C_1e^t\cos t+C_2e^t\sin t+C_3e^t\cos t+C_4e^t\sin t+C_5e^{3t}$

B. $y=C_1e^{2t}\cos t+C_2e^{2t}\sin t+C_3te^{2t}\cos t+C_4te^{2t}\sin t+C_5e^{3t}$

C. $y=C_1e^{-t}\cos 2t+C_2e^{-t}\sin 2t+C_3te^{-t}\cos 2t+C_4te^{-t}\sin 2t+C_5e^{3t}$

D. $y=C_1e^{-t}+C_2te^{-t}+C_3t^2e^{-t}+C_4t^3e^{-t}+C_5e^{-3t}$

E. $y=C_1e^{-t}\cos t+C_2e^{-t}\sin t+C_3e^{3t}$

(20 pts) **3.** Find the general solution of $y'' + 4y' = 8t + 10$.

Complementary soln: $y'' + 4y' = 0 \Rightarrow r^2 + 4r = 0$

$r(r+4) = 0$

$$\boxed{y_c = C_1 + C_2 e^{-4t}}$$

Particular soln: $y_p = t^s[At + B] = At^2 + Bt$

$s = \cancel{0}, 1, 2$

so $y_p' = 2At + B$

$y_p'' = 2A$

so $y_p'' + 4y_p' = 8t + 10$

$2A + 4(2At + B) = 8t + 10$

$(8A)t + (2A + 4B) = 8t + 10$

$\Rightarrow \begin{cases} 8A = 8 \\ 2A + 4B = 10 \end{cases} \Rightarrow A = 1, \quad B = 2$

$\therefore y_p = t^2 + 2t$

$$y = \boxed{C_1 + C_2 e^{-4t} + (t^2 + 2t)}$$

(5 pts) **4.** A 16 pound object attached to a spring stretches it 2 ft. If the object is pulled down 18 inches below equilibrium and then given an upward velocity of 3 ft/sec and there is no damping, the initial value problem for the vertical displacement $u(t)$ is

$mu'' + \gamma u' + ku = F(t)$

$\gamma = 0$ $\quad 16 = mg = m(32)$

$F(t) = 0$ $\quad m = \frac{1}{2}$

@ equilibrium $F_s = F_g$

$kd = mg$

$k(2) = 16$

$k = 8$

A. $2u'' + 8u = 0, \quad u(0) = \frac{3}{2},\ u'(0) = 3$

B. $\frac{1}{2}u'' + 8u = 0, \quad u(0) = \frac{3}{2},\ u'(0) = -3$

C. $\frac{1}{2}u'' + 8u = 0, \quad u(0) = 2,\ u'(0) = 3$

D. $2u'' + 16u = 0, \quad u(0) = \frac{3}{2},\ u'(0) = 3$

E. $16u'' + 8u = 0, \quad u(0) = \frac{3}{2},\ u'(0) = -3$

$$\therefore \begin{cases} \frac{1}{2}u'' + 8u = 0 \\ u(0) = \frac{18}{12} = \frac{3}{2}, \quad u'(0) = -3 \end{cases}$$

(10 pts) **5.** Find the Laplace transform of $f(t) = (e^{3t} - t^2)^2$.

$$f(t) = e^{6t} - 2t^2e^{3t} + t^4$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{6t}\} - 2\,\mathcal{L}\{t^2e^{3t}\} + \mathcal{L}\{t^4\}$$

$$= \frac{1}{s-6} - 2\left(\frac{2!}{(s-3)^3}\right) + \frac{4!}{s^5}$$

$$\mathcal{L}\{f(t)\} = \boxed{\frac{1}{s-6} - \frac{4}{(s-3)^3} + \frac{24}{s^5}}$$

(5 pts) **6.** Find the Laplace transform of the solution of $\begin{cases} 2y'' - 3y = 10\,t \\ y(0) = 4 \\ y'(0) = 0 \end{cases}$.

(*Do not solve the initial value problem*)

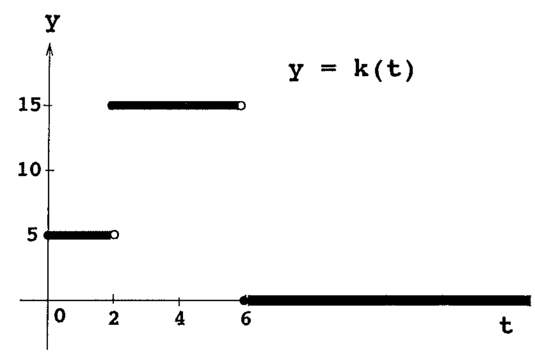
$2\mathcal{L}\{y''\} - 3\,\mathcal{L}\{y\} = 10\,\mathcal{L}\{t\}$

$2\left[s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{=4} - \underbrace{\cancel{y'(0)}}_{=0}\right] - 3\,\mathcal{L}\{y\} = \frac{10}{s^2}$

$\mathcal{L}\{y\}(2s^2 - 3) - 8s = \frac{10}{s^2}$

$\mathcal{L}\{y\} = Y(s) = \boxed{\dfrac{\frac{10}{s^2} + 8s}{(2s^2 - 3)} = \dfrac{10 + 8s^3}{s^2(2s^2 - 3)}}$

(5 pts) **7.** Find $\mathcal{L}\{k(t)\}$ if $k(t)$ is as shown:



$k(t) = 5\left[u_0(t) - u_2(t)\right] + 15\left[u_2(t) - u_6(t)\right]$

$= 5\left[1 - u_2(t)\right] + 15\,u_2(t) - 15\,u_6(t)$

$= 5 + 10\,u_2(t) - 15\,u_6(t)$

$\mathcal{L}\{k(t)\} = \boxed{\dfrac{5}{s} + \dfrac{10e^{-2s}}{s} - \dfrac{15e^{-6s}}{s}}$

(5 pts) **8.** The Laplace transform of $f(t) = u_2(t)(3t+2)$ is

A. $e^{-2s}\left(\frac{3}{s^2}+\frac{2}{s}\right)$

B. $e^{-2s}\left(\frac{3}{s^2}-\frac{2}{s}\right)$

C. $e^{-2s}\left(\frac{3}{s^2}-\frac{8}{s}\right)$

D. $e^{-2s}\left(\frac{3}{s^2}+\frac{8}{s}\right)$

E. $\frac{e^{-2s}}{s}\left(\frac{3}{s^2}+\frac{2}{s}\right)$

$$f(t) = u_2(t)(3t+2) = u_2(t)\underbrace{\left[3(t-2)+8\right]}_{g(t-2)}$$

so $g(t) = 3t+8$

$$\mathcal{L}\{f(t)\} = e^{-2s}\left(\frac{3}{s^2}+\frac{8}{s}\right)$$

(5 pts) **9.** Find the inverse Laplace transform of $F(s) = \dfrac{4s-4}{s^2+2s+10}$.

A. $f(t) = 4\,e^{t}\cos 3t$

B. $f(t) = 4\,e^{-t}\cos 3t - \frac{4}{3}\,e^{-t}\sin 3t$

C. $f(t) = 4\,e^{-t}\cos 3t - \frac{8}{3}\,e^{-t}\sin 3t$

D. $f(t) = 4\,e^{-t}\cos 3t - 4\,e^{-t}\sin 3t$

E. $f(t) = 3\,e^{-9t} - e^{-t}$

$$F(s) = \frac{4s-4}{s^2+2s+10} = \frac{4(s+1)-8}{(s+1)^2+9}$$

$$\mathcal{L}^{-1}\{F(s)\} = 4\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+9}\right\} - \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$= 4e^{-t}\cos 3t - \frac{8}{3}e^{-t}\sin 3t$$

(10 pts) **10.** Find the inverse Laplace Transform of $F(s) = \dfrac{12e^{-8s}}{s(s^2+4)}$.

$$\mathcal{L}^{-1}\left\{e^{-8s}\left[\frac{12}{s(s^2+4)}\right]\right\} = u_8(t)\,f(t-8),$$

where $f(t) = \mathcal{L}^{-1}\left\{\dfrac{12}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{3}{s} + \dfrac{(-3s)}{s^2+4}\right\}$

by partial fractions

$$= 3 - 3\cos 2t$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = u_8(t)\left[3 - 3\cos 2(t-8)\right]$$

$$\mathcal{L}^{-1}\left\{\frac{12e^{-8s}}{s(s^2+4)}\right\} = \boxed{u_8(t)\left[3 - 3\cos 2(t-8)\right]}$$

(20 pts) **11.** Find the solution of the initial value problem : $\begin{cases} y'' - 4y = 8 + 8u_3(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$\mathcal{L}\{y''\} - 4\,\mathcal{L}\{y\} = \mathcal{L}\{8\} + 8\,\mathcal{L}\{u_3(t)\}$$

$$s^2\mathcal{L}\{y\} - s\underbrace{\cancel{y(0)}}_{=0} - \underbrace{\cancel{y'(0)}}_{=0} - 4\,\mathcal{L}\{y\} = \frac{8}{s} + \frac{8e^{-3s}}{s}$$

$$\mathcal{L}\{y\}(s^2-4) = \frac{8}{s} + \frac{8e^{-3s}}{s} \Rightarrow \mathcal{L}\{y\} = \frac{8}{s(s^2-4)} + e^{-3s}\left(\frac{8}{s(s^2-4)}\right)$$

$$\Rightarrow y = \mathcal{L}^{-1}\left\{\frac{8}{s(s^2-4)}\right\} + \mathcal{L}^{-1}\left\{\frac{8e^{-3s}}{s(s^2-4)}\right\}$$

Now since $\dfrac{8}{s(s^2-4)} = \dfrac{-2}{s} + \dfrac{1}{s-2} + \dfrac{1}{s+2}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{8}{s(s^2-4)}\right\} = -2 + e^{2t} + e^{-2t} \quad (\text{or } -2 + 2\cosh 2t)$$

$$\therefore \boxed{y = [-2 + e^{2t} + e^{-2t}] + u_3(t)[-2 + e^{2(t-3)} + e^{-2(t-3)}]}$$

or, $\boxed{y = [-2 + 2\cosh 2t] + u_3(t)[-2 + 2\cosh 2(t-3)]}$

$y =$

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	t^n	$\dfrac{n!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \ c > 0$
16.	$\displaystyle\int_0^t f(t-\tau)\, g(\tau)\, d\tau$	$F(s)\, G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$