MA 266
Review Topics - Exam # 2 (updated)

(1) **First Order Differential Equations.** (Separable, 1st Order Linear, Homogeneous, Exact)

(2) **Second Order Linear Homogeneous with Equations Constant Coefficients.**

The differential equation \( ay'' + by' + cy = 0 \) has Characteristic Equation \( ar^2 + br + c = 0 \). Call the roots \( r_1 \) and \( r_2 \). The general solution of \( ay'' + by' + cy = 0 \) is as follows:

(a) If \( r_1, r_2 \) are real and distinct \( \Rightarrow y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \)

(b) If \( r_1 = \lambda + i\mu \) (hence \( r_2 = \lambda - i\mu \)) \( \Rightarrow y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t \)

(c) If \( r_1 = r_2 \) (repeated roots) \( \Rightarrow y = C_1 e^{r_1 t} + C_2 te^{r_1 t} \)

(3) **Theory of 2nd Linear Order Equations.**

Wronskian of \( y_1, y_2 \) is \( W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} \).

(a) The functions \( y_1(t) \) and \( y_2(t) \) are linearly independent over \( a < t < b \) if \( W(y_1, y_2) \neq 0 \) for at least one point in the interval.

(b) **THEOREM (Existence & Uniqueness)** If \( p(t), q(t) \) and \( g(t) \) are continuous in an open interval \( \alpha < t < \beta \) containing \( t_0 \), then the IVP \( \begin{cases} y'' + p(t) y' + q(t) y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases} \)

has a unique solution \( y = \phi(t) \) defined in the open interval \( \alpha < t < \beta \).

(c) **Superposition Principle** If \( y_1(t) \) and \( y_2(t) \) are solutions of the 2nd order linear homogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = 0 \) over the interval \( a < t < b \), then \( y = C_1 y_1(t) + C_2 y_2(t) \) is also a solution for any constants \( C_1 \) and \( C_2 \).

(d) **THEOREM (Homogeneous)** If \( y_1(t) \) and \( y_2(t) \) are solutions of the linear homogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = 0 \) in some interval \( I \) and \( W(y_1, y_2) \neq 0 \) for some \( t_1 \) in \( I \), then the general solution is \( y_c(t) = C_1 y_1(t) + C_2 y_2(t) \). This is usually called the *complementary solution* and we say that \( y_1(t), y_2(t) \) form a Fundamental Set of Solutions (FSS) to the differential equation.

(e) **THEOREM (Nonhomogeneous)** The general solution of the nonhomogeneous equation

\[ P(t)y'' + Q(t)y' + R(t)y = G(t) \]

is \( y(t) = y_c(t) + y_p(t) \), where \( y_c(t) = C_1 y_1(t) + C_2 y_2(t) \) is the general solution of the corresponding homogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = 0 \) and \( y_p(t) \) is a particular solution of the nonhomogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = G(t) \).

(f) **Useful Remark**: If \( y_{p_1}(t) \) is a particular solution of \( P(t)y'' + Q(t)y' + R(t)y = G_1(t) \) and if \( y_{p_2}(t) \) is a particular solution of \( P(t)y'' + Q(t)y' + R(t)y = G_2(t) \), then

\[ y_p(t) = y_{p_1}(t) + y_{p_2}(t) \]

is a particular solution of \( P(t)y'' + Q(t)y' + R(t)y = [G_1(t) + G_2(t)] \).
(4) **Reduction of Order.** If \( y_1(t) \) is one solution of \( P(t)y'' + Q(t)y' + R(t)y = 0 \), then a second solution may be obtained using the substitution \( y = v(t)y_1(t) \). This reduces the original 2\(^{nd}\) order equation to a 1\(^{st}\) equation using the substitution \( w = \frac{dv}{dt} \). Solve that first order equation for \( w \), then since \( w = \frac{dv}{dt} \), solve this 1\(^{st}\) order equation to determine the function \( v \).

(5) **Finding A Particular Solution \( y_p(t) \) to Nonhomogeneous Equations.**

You can always use the method of Variation of Parameters to find a particular solution \( y_p(t) \) of the linear nonhomogeneous equation \( y'' + p(t)y' + q(t)y = g(t) \). Variation of Parameters may require integration techniques.

If the coefficients of the differential equation are \textbf{constants} rather than functions and if \( g(t) \) has a very special form (see table below), it is usually easier to use Undetermined Coefficients:

(a) **Undetermined Coefficients** - IF \( ay'' + by' + cy = g(t) \) AND \( g(t) \) is as below:

<table>
<thead>
<tr>
<th>( g(t) )</th>
<th>Form of ( y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_m(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0 )</td>
<td>( t^s { A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0 } )</td>
</tr>
<tr>
<td>( e^{at} P_m(t) )</td>
<td>( t^s { e^{at} (A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0) } )</td>
</tr>
</tbody>
</table>
| \( e^{at} P_m(t) \cos \beta t \) or \( e^{at} P_m(t) \sin \beta t \) | \( t^s \{ e^{at} [F_m(t) \cos \beta t + G_m(t) \sin \beta t] \} \)  

where \( s = \) the smallest nonnegative integer (\( s = 0,1 \) or 2) such that no term of \( y_p(t) \) is a solution of the corresponding homogeneous equation. In other words, no term of \( y_p(t) \) is a term of \( y_c(t) \). (\( F_m(t) \), \( G_m(t) \) are both polynomials of degree \( m \).)

(b) **Variation of Parameters** - If \( y_1(t) \) and \( y_2(t) \) are two independent solutions of the homogeneous equation \( y'' + p(t)y' + q(t)y = 0 \), then a particular solution \( y_p(t) \) of the nonhomogeneous equation

\[ y'' + p(t)y' + q(t)y = g(t) \quad (\ast) \]

has the form

\[ y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \]

where

\[
\begin{align*}
  u'_1 &= \begin{vmatrix}
    0 & y_2 \\
    g(t) & y'_2 \\
    y_1 & y'_2 \\
  \end{vmatrix},
  u'_2 &= \begin{vmatrix}
    y_1 & 0 \\
    y'_1 & g(t) \\
    y_1 & y'_2 \\
  \end{vmatrix}.
\end{align*}
\]

\textbf{Remember:} Coefficient of \( y'' \) in \((\ast)\) must be “1” in order to use the above formulas.
Spring-Mass Systems

\[
\begin{align*}
    m u'' + \gamma u' + k u &= F(t) \\
    u(0) &= u_0, \quad u'(0) = u_1
\end{align*}
\]

- \(m\) = mass of object, \(\gamma\) = damping constant, \(k\) = spring constant, \(F(t)\) = external force
- Weight \(w = mg\)

Hooke’s Law: \(F_s = k d\),

\[
equilibrium: \quad F_s = F_g \quad kd = mg
\]

**Undamped Free Vibrations:** \(m u'' + k u = 0\) (Simple Harmonic Motion)

Note that \(A \cos \omega_0 t + B \sin \omega_0 t = R \cos(\omega_0 t - \delta)\), where \(R = \sqrt{A^2 + B^2}\) = amplitude,
- \(\omega_0\) = frequency, \(\frac{2\pi}{\omega_0}\) = period and \(\delta\) = phase shift determined by \(\tan \delta = \frac{B}{A}\).

**Damped Free Vibrations:** \(m u'' + \gamma u' + k u = 0\)

- (i) \(\gamma^2 - 4km > 0\) (overdamped) \(\iff\) distinct real roots to CE
- (ii) \(\gamma^2 - 4km = 0\) (critically damped) \(\iff\) repeated roots to CE
- (iii) \(\gamma^2 - 4km < 0\) (underdamped) \(\iff\) complex roots to CE (motion is oscillatory)

**Forced Vibrations:** \((F(t) = F_0 \cos \omega t)\) or \((F(t) = F_0 \sin \omega t)\), for example

- (i) \(m u'' + \gamma u' + k u = F(t)\) (Damped) In this case if you write the general solution as \(u(t) = u_T(t) + u_\infty(t)\), then \(u_T(t) = Transient \text{ Solution}\) (i.e. the part of \(u(t)\) such that \(u_T(t) \to 0\) as \(t \to \infty\)) and \(u_\infty(t)\) = Steady-State Solution (the solution behaves like this function in the long run).

- (ii) \(m u'' + k u = F_0 \cos \omega t\) (Undamped) If \(\omega = \omega_0 = \sqrt{\frac{k}{m}}\) \(\Rightarrow\) Resonance occurs and the solution is unbounded; while if \(\omega \neq \omega_0\) then motion is a series of beats (solution is bounded)

**n\textsuperscript{th} Order Linear Homogeneous Equations With Constant Coefficients**

\[
a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \quad (*)
\]

This differential equation has \(n\) independent solutions.

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**Characteristic Equation:** \(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0\) will have \(n\) characteristic roots that may be real and distinct, repeated, complex, or complex and repeated.
(a) For each real root \( r \) that is not repeated \( \Rightarrow \) get a solution of (*) \( e^{rt} \)

(b) For each real root \( r \) that is repeated \( m \) times \( \Rightarrow \) get \( m \) independent solutions of (*):
\[ e^{rt}, \ te^{rt}, \ t^2 e^{rt}, \ldots, t^{m-1} e^{rt} \]

(c) For each complex root \( r = \lambda + i\mu \) repeated \( m \) times \( \Rightarrow \) get \( 2m \) solutions of (*):
\[ e^{\lambda t} \cos \mu t, \ te^{\lambda t} \cos \mu t, \ldots, t^{m-1} e^{\lambda t} \cos \mu t \]
\[ \text{and} \]
\[ e^{\lambda t} \sin \mu t, \ te^{\lambda t} \sin \mu t, \ldots, t^{m-1} e^{\lambda t} \sin \mu t \]

(don’t need to consider its conjugate root \( \lambda - i\mu \))

(8) **Undetermined Coefficients for \( n \)th Order Linear Equations**

This can only be used to find \( y_p(t) \) of \( a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \) and \( g(t) \) one of the 3 very SPECIAL FORMS in table in (5) above. The particular solution has the same form as before:
\[ y_p(t) = t^s [\cdots] \] where \( s \) = the smallest nonnegative integer such that no term of \( y_p(t) \) is a term of \( y_c(t) \), except this time \( s = 0, 1, 2, \ldots, n \).

(9) **Laplace Transforms**

(a) Be able to compute Laplace transforms using definition:
\[ \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \, dt \]

and using a table of Laplace transforms (see table on page 317) and using linearity: \( \mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \), \( \mathcal{L}\{cf(t)\} = c \mathcal{L}\{f(t)\} \).

(b) **Computing Inverse Laplace Transforms**: Must be able to use a table of Laplace transforms usually together with Partial Fractions or Completing the Square, to find inverse Laplace transforms:
\[ f(t) = \mathcal{L}^{-1}\{F(s)\} \]

(c) **Solving Initial Value Problems**: Recall that
\[ \mathcal{L}\{y'\} = s \mathcal{L}\{y\} - y(0) \]
\[ \mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - s y(0) - y'(0) \]
\[ \mathcal{L}\{y'''\} = s^3 \mathcal{L}\{y\} - s^2 y(0) - s y'(0) - y''(0) \]
\[ \vdots \]

(d) **Discontinuous Functions**:

(i) **Unit Step Function** (Heaviside Function): If \( c \geq 0 \), \( u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \)

\[ y_c(t) \]

\[ \mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \]
(ii) **Unit “Pulse” Function:** \( u_a(t) - u_b(t) = \begin{cases} 1, & a \leq t < b \\ 0, & \text{otherwise} \end{cases} \)

![Graph of \( y = u_a(t) - u_b(t) \)]

(iii) **Translated Functions:** \( y = g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases} = u_c(t)f(t-c) \).

![Graphs of \( y = f(t) \) and \( y = g(t) \)]

\[
\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s), \text{ where } F(s) = \mathcal{L}\{f(t)\}
\]

Thus,
\[
\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c), \text{ where } f(t) = \mathcal{L}^{-1}\{F(s)\}
\]

A useful formula **NOT** in the book:
\[
\mathcal{L}\{u_c(t)h(t)\} = e^{-cs}\mathcal{L}\{h(t+c)\}
\]

(iv) **Unit Impulse Functions:** If \( y = \delta(t-c) \) (\( c \geq 0 \)), then
\[
\mathcal{L}\{\delta(t-c)\} = e^{-cs}
\]

(e) **Convolutions:**
\[
\mathcal{L}\{(f \ast g)(t)\} = \mathcal{L}\{\int_0^t f(t-\tau)g(\tau)\,d\tau\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}
\]
(10) Systems of Linear Differential Equations: \( x'(t) = Ax(t) \)

(a) Rewrite a single \( n^{th} \) order equation \( p_0(t)y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t) \) as a system of 1st order equations. Use the substitution :

\[
x_1 = y \\
x_2 = y' \\
\vdots \\
x_n = y^{(n-1)}
\]

Let to get 1st Order System:

\[
\begin{cases}
x_1' = x_2 \\
x_2' = x_3 \\
\vdots \\
x_{n-1}' = x_n \\
x_n' = \frac{1}{p_0} \left\{ -p_n x_1 - p_{n-1} x_2 - \cdots - p_1 x_n + g(t) \right\}
\end{cases}
\]

(b) Existence & Uniqueness Theorem for Systems. If \( P(t) \) and \( g(t) \) are continuous on an interval \( \alpha < t < \beta \) containing \( t_0 \), then the IVP \( \begin{cases} x'(t) = P(t)x(t) \\ x(t_0) = x_0 \end{cases} \) has a unique solution \( x(t) \) defined on the interval \( \alpha < t < \beta \).

(c) The set of vectors \( \{x^{(1)}, x^{(2)}, \ldots, x^{(m)}\} \) is linearly independent if the equation

\[
k_1x^{(1)} + k_2x^{(2)} + \cdots + k_mx^{(m)} = 0
\]

is satisfied only for \( k_1 = k_2 = \cdots = k_m = 0 \). This means you cannot write any one of these vectors as a linear combination of the others.

(d) Solve 2 \times 2 \) systems of 1st order equations

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

i.e., \( x' = Ax \) using :

(i) Elimination Method: Basic idea - eliminate one of the unknowns (either \( x_1 \) or \( x_2 \)) from the original system to get an equivalent single 2nd order differential equation.

(ii) Eigenvalues & Eigenvectors Method: See (11) below for solutions via this method and corresponding phase portraits.

Eigenvalue: If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then the eigenvalues of \( A \) are the roots of

\[
|A - \lambda I| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0
\]

Eigenvector: \( \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is a nonzero solution to \( (A - \lambda I) \vec{v} = \vec{0} \).

(e) If \( x^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix} \) and if \( x^{(2)}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix} \), then the Wronskian is

\[
W[x^{(1)}, x^{(2)}] = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}.
\]

If \( x^{(1)}(t) \) and \( x^{(2)}(t) \) are solutions of \( x' = Ax \) and \( W[x^{(1)}, x^{(2)}](t_1) \neq 0 \), then the set \( \{x^{(1)}(t), x^{(2)}(t)\} \) forms a Fundamental Set of Solutions of the system and a Fundamental Matrix is

\[
\Phi(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}.
\]
The following describes how to find the general solution to (*) and plot solutions (trajectories). A plot of the trajectories of a given homogeneous system
\[ x' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x \quad (*) \]
is called a phase portrait. To sketch the phase portrait, we need to find the corresponding eigenvalues and eigenvectors of the matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and then consider 3 cases:

(a) \( \lambda_1 < \lambda_2 \), real and distinct: If \( v^{(1)} \), \( v^{(2)} \) are e-vectors corresponding to \( \lambda_1 \) and \( \lambda_2 \), respectively \( \Rightarrow x^{(1)}(t) = e^{\lambda_1 t} v^{(1)} \) and \( x^{(2)}(t) = e^{\lambda_2 t} v^{(2)} \) are solutions and hence general solution of (*) is \( x(t) = C_1 x^{(1)}(t) + C_2 x^{(2)}(t) \) and hence if \( \lambda_1 < \lambda_2 \):
\[
 x(t) = \underbrace{C_1 e^{\lambda_1 t} v^{(1)}}_{\text{dominates}} + \underbrace{C_2 e^{\lambda_2 t} v^{(2)}}_{\text{dominates}} \quad \text{as } t \to -\infty \quad \text{as } t \to \infty
\]
(b) $\lambda_1 = \alpha + i \beta$:  If $w = a + i b$ is a complex e-vector corresponding to $\lambda_1$ then $\Rightarrow$

$x^{(1)}(t) = \Re \{ e^{\lambda_1 t} w \} = e^{\alpha t} (a \cos \beta t - b \sin \beta t)$ and

$x^{(2)}(t) = \Im \{ e^{\lambda_1 t} w \} = e^{\alpha t} (a \sin \beta t + b \cos \beta t)$ are real-valued solutions and hence general solution of ($\ast$) is $x(t) = C_1 x^{(1)}(t) + C_2 x^{(2)}(t)$.

If say $\alpha < 0$:

(Test a point to decide which)

(c) $\lambda_1 = \lambda_2$:  If there is only one linearly independent eigenvector corresponding to $\lambda_1$, then solutions to $\mathbf{x}' = A \mathbf{x}$ are

$x^{(1)}(t) = e^{\lambda_1 t} \mathbf{v}$ and $x^{(2)}(t) = t e^{\lambda_1 t} \mathbf{v} + e^{\lambda_1 t} \mathbf{a}$, where

$$(A - \lambda_1 I) \mathbf{v} = 0$$

$$(A - \lambda_1 I) \mathbf{a} = \mathbf{v}$$

($\mathbf{v}$ is an eigenvector of $A$, while $\mathbf{a}$ is called a “generalized eigenvector” of $A$)

The general solution of the system ($\ast$) is $x(t) = C_1 x^{(1)}(t) + C_2 x^{(2)}(t)$ and hence:

$$x(t) = C_1 e^{\lambda_1 t} \mathbf{v} + C_2 \left[ t e^{\lambda_1 t} \mathbf{v} + e^{\lambda_1 t} \mathbf{a} \right]$$

dominates as $t \rightarrow \pm \infty$

If say $\lambda_1 < 0$:

(Test a point to decide which)
(12) Particular Solutions to Nonhomogeneous Linear Systems :

\[ x' = A \mathbf{x} + g(t) \]

(a) **Undetermined Coefficients for Systems** The column vector \( \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \) must have each component function \( g_1(t) \) and \( g_2(t) \) as one of the three special forms like those for Undetermined Coefficients for regular 2nd order equations and \( A \) must be a constant matrix. The main difference is if say \( g(t) = u e^{\lambda t} \) and \( \lambda \) is also an eigenvalue of \( A \), then try a particular solution of the form \( x_p = a t e^{\lambda t} + b e^{\lambda t} \).

(b) **Variation of Parameters for Systems** : \( x' = A(t) \mathbf{x} + g(t) \):

\[ x_p(t) = \Phi(t) \int \Phi^{-1}(t) \mathbf{g}(t) \, dt, \]

where \( \Phi(t) \) is a Fundamental Matrix of the homogeneous system \( x' = A(t) \mathbf{x} + g(t) \) can have any form and \( A \) need not be a constant matrix.

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**Practice Problems**

[1] For what value of \( \alpha \) will the solution to the IVP \( \begin{cases} y'' - y' - 2y = 0 \\ y(0) = \alpha \\ y'(0) = 2 \end{cases} \) satisfy \( y \to 0 \) as \( t \to \infty \)?

[2] (a) Show that \( y_1 = x \) and \( y_2 = x^{-1} \) are solutions of the differential equation \( x^2 y'' + xy' - y = 0 \). (b) Evaluate the Wronskian \( W(y_2, y_1) \) at \( x = \frac{1}{2} \). (c) Find the solution of the initial value problem \( x^2 y'' + xy' - y = 0 \), \( y(1) = 2 \), \( y'(1) = 4 \).

[3] Find the largest open interval for which the initial value problem \( 3x^2 y'' + y' + \frac{1}{x-2} y = \frac{1}{x-3} \), \( y(1) = 3 \), \( y'(1) = 2 \), has a solution.

In Problems 4, 5, and 6 find the general solution of the homogeneous differential equations in (a) and use the method of **Undetermined Coefficients** to find a particular solution \( y_p \) in (b) and find the form of a particular solution (c).

[4] (a) \( y'' - 5y' + 6y = 0 \) (b) \( y'' - 5y' + 6y = t^2 \) (c) \( y'' - 5y' + 6y = e^{2t} + \cos(3t) \)

[5] (a) \( y'' - 6y' + 9y = 0 \) (b) \( y'' - 6y' + 9y = te^{3t} \) (c) \( y'' - 6y' + 9y = e^t + \cos(3t) \)

[6] (a) \( y'' - 2y' + 10y = 0 \) (b) \( y'' - 2y' + 10y = e^x + \cos(3x) \) (c) \( y'' - 2y' + 10y = e^x \cos(3x) \)

[7] Find the general solution to (a) \( y'' + y' - 6y = 7e^{4t} \) (b) \( y'' + y' - 6y = 7e^{4t} - 100 \sin t \)

[8] Solve this IVP: \( y'' - y' = 4t \), \( y(0) = 0 \), \( y'(0) = 0 \).

[9] Find the general solution to \( y'' + y' = \tan t \), \( 0 < x < \frac{\pi}{2} \).

[10] The differential equation \( x^2 y'' - 2xy' + 2y = 0 \) has solutions \( y_1(x) = x \) and \( y_2(x) = x^2 \). Use the method of **Variation of Parameters** to find a solution of \( x^2 y'' - 2xy' + 2y = 2x^2 \).

[11] The differential equation \( x^2 y'' + xy' - y = 0 \) has one solution \( y_1(x) = x \). Use the method of **Reduction of Order** to find a second (linearly independent) solution of \( x^2 y'' + xy' - y = 0 \).

[12] For what nonnegative values of \( \gamma \) will the solution of the initial value problem \( u'' + \gamma u' + 4u = 0 \), \( u(0) = 4 \), \( u'(0) = 0 \) oscillate?

[13] (a) For what positive values of \( k \) does the solution of the initial value problem \( 2u'' + ku = 3 \cos(2t) \), \( u(0) = 0 \), \( u'(0) = 0 \), become unbounded (Resonance)?
Find the general solution of the differential equation
$$2u'' + u' + ku = 3\cos(2t), \quad u(0) = 0, \quad u'(0) = 0,$$ become unbounded (Resonance)?

[14] Find the steady-state solution of the IVP $y'' + 4y' + 4y = \sin t$, $y(0) = 0$, $y'(0) = 0$.

[15] A 4-kg mass stretches a spring 0.392 m. If the mass is released from 1 m below the equilibrium position with a downward velocity of 10 m/sec, what is the maximum displacement?

In Problems 16 and 17 find the general solution of the homogeneous differential equations in (a) and use the method of Undetermined Coefficients to find the form of a particular solution of the nonhomogeneous equation in (b).

[16] (a) $y''' - y' = 0$ \quad (b) $y''' - y' = t + e^t$

[17] (a) $y''' - y'' - y' + y = 0$ \quad (b) $y''' - y'' + y = e^t + \cos t$

[18] Find the solution of the initial value problem $y''' - 2y'' + y' = 0$, $y(0) = 2$, $y'(0) = 0$, $y''(0) = 1$.

[19] Find the general solution of the differential equation $y'' + y' = t^2$.

[20] Find the general solution of $y'' + 4y' = -10\cos 2t$.

[21] Find a fundamental set of solutions of $y^{(5)} - 4y''' = 0$.

[22] Find the Laplace transform of these functions:
(a) $f(t) = 3 - e^{2t}$ \quad (b) $g(t) = 100t^5$ \quad (c) $h(t) = \cosh \pi t$ \quad (d) $k(t) = -10t^3e^{5t}$

[23] Find the inverse Laplace transform of these functions:
(a) $F(s) = \frac{9}{s^2 - s - 2}$ \quad (b) $F(s) = \frac{s}{(s - 1)^2}$ \quad (c) $F(s) = \frac{8}{(s + 1)^4}$ \quad (d) $F(s) = \frac{3s + 2}{s^2 + 2s + 5}$

[24] Solve these initial value problems:
(a) \begin{align*}
  y'' - y' - 6y &= 0 \\
  y(0) &= 1 \\
  y'(0) &= -1
\end{align*}

(b) \begin{align*}
  y'' - 2y' + 2y &= \cos t \\
  y(0) &= 1 \\
  y'(0) &= 0
\end{align*}

(c) $y'' - y = \begin{cases} 1, & t < 5 \\
                           2, & 5 \leq t < \infty \end{cases}$; $y(0) = y'(0) = 0$.

(d) $y'' + 4y = \begin{cases} t, & t < 1 \\
                          0, & 1 \leq t < \infty \end{cases}$; $y(0) = y'(0) = 0$.

(e) $y' + y = g(t)$, $g(0) = 0$ and where $g(t)$:

\[ \begin{array}{c}
  y = g(t) \\
  \text{value 3} \\
  \text{value 0} \\
  \text{value 2} \\
  \text{value 4} \\
  \text{value t}
\end{array} \]

(f) $y'' + 4y = \delta(t - 3)$, $y(0) = y'(0) = 0$

[25] $\mathcal{L} \left\{ \int_0^t 100e^{-2\tau}\cos \pi(t - \tau) \, d\tau \right\} = ?$

[26] If $g(t) = \mathcal{L}^{-1}\{G(s)\}$, then $\mathcal{L}^{-1}\left\{ \frac{G(s)}{(s-3)^2} \right\} = ?$

[27] Use the Elimination Method to solve the system
\[ \begin{cases} x'_1 = x_1 + x_2 \\
                   x'_2 = 4x_1 + x_2 \end{cases} \]

[28] Rewrite the $2^{nd}$ order differential equation $y'' + 2y' + 3ty = \cos t$ with $y(0) = 1$, $y'(0) = 4$ as a system of $1^{st}$ order differential equations.

[29] Find eigenvalues and corresponding eigenvectors of
(a) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ \quad (b) $A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$
[30] Find the solution of the IVP \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \)

Find a fundamental matrix \( \Phi(t) \).

[31] Solve \( \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \)

[32] Find the general solution of the system \( \vec{x}'(t) = A \vec{x}(t) \), where \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

[33] Tank #1 initially holds 50 gals of brine with a concentration of 1 lb/gal, while Tank #2 initially holds 25 gals of brine with a concentration of 3 lb/gal. Pure H\(_2\)O flows into Tank #1 at 5 gal/min. The well-stirred solution from Tank #1 then flows into Tank #2 at 5 gal/min. The solution in Tank #2 flows out at 5 gal/min. Set up and solve an IVP that gives \( x_1(t) \) and \( x_2(t) \), the amount of salt in Tanks #1 and #2, respectively, at time \( t \).

[34] Tank #1 initially holds 50 gals of brine with concentration of 1 lb/gal and Tank #2 initially holds 25 gals of brine with concentration 3 lb/gal. The solution in Tank #1 flows at 5 gal/min into Tank #2, while the solution in Tank #2 flows back into Tank #1 at 5 gal/min. Set up an IVP that gives \( x_1(t) \) and \( x_2(t) \), the amount of salt in Tanks #1 and #2, respectively, at time \( t \).

[35] Find the general solution of \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t. \)

[36] Find a particular solution of \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \)

[37] Find the general solution of \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 6e^{-t} \\ 1 \end{pmatrix}. \)

[38] Match the phase portraits shown below that best corresponds to each of the given systems of differential equations:

(i) \( \vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} \); Solution: \( \vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} \)
(ii) \( \vec{x}' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \vec{x} \); Solution: \( \vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} \)
(iii) \( \vec{x}' = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \vec{x} \); Solution: \( \vec{x}(t) = C_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^t + C_2 e^t \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \)
(iv) \( \vec{x}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{x} \); Solution: \( \vec{x}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t} \)

(A) (B)
[1] \( \alpha = -2 \)  
[2] (b) \( W(x^{-1}, x)(\frac{1}{2}) = 4 \); (c) \( y = 3x - x^{-1} \)  
[3] \( 0 < x < 2 \)  
[4] (a) \( y = C_1e^{2t} + C_2e^{3t} \) (b) \( y = At^2 + Bt + C \) (c) \( y = Ate^{2t} + B \cos(3t) + C \sin(3t) \)  
[5] (a) \( y = C_1e^{3t} + C_2e^{2t} \) (b) \( y = t^2(At + B)e^{3t} \) (c) \( y = Ae^t + B \cos(3t) + C \sin(3t) \)  
[6] (a) \( y = C_1e^x + C_2e^{3x} \) (b) \( y = A \cos(3x) + C \sin(3x) \)  
(c) \( y = x(A \cos(3x) + B \sin(3x))e^x \)  
[7] (a) \( y = C_1e^{-3t} + C_2e^{2t} + \frac{1}{2}e^{4t} \) (b) \( y = C_1e^{-3t} + C_2e^{2t} + \frac{1}{2}e^{4t} + 2 \cos t + 14 \sin t \)  
[8] \( y = -4 + 4e^t - 2t^2 - 4t \)  
[9] \( y = C_1 \cos t + C_2 \sin t - (\cos t) \ln(\sec t + \tan t) \)  
[10] \( y = 2x^2 \ln x \) or \( y = 2x^2 \ln x + (C_1x + C_2x^2) \)  
[11] \( y = x^{-1} \) or \( y = Ax^{-1} + Bx, A \neq 0 \)  
[12] \( 0 \leq \gamma < 4 \)  
[13] (a) \( k = 8 \) (resonance) (b) NO value of \( k \), all solutions are bounded.  
[14] \( y = \frac{1}{2} \sin(3 \sin t - 4 \cos t) \)  
[15] \( u(t) = \cos 5t + 2 \sin 5t = \sqrt{5} \cos(5t - \delta), \delta = \tan^{-1}2 \approx 1.1 \) Thus amplitude = \( \sqrt{5} \).  
[16] (a) \( y = C_1 + C_2e^{-t} + C_3e^t \) (b) \( y = t(At + B) + Cte^t \)  
[17] (a) \( y = C_1e^t + C_2te^t + C_3e^{-t} \) (b) \( y = At^2e^t + B \cos t + C \sin t \)  
[18] \( y = 3 - e^t + te^t \)  
[19] \( y = C_1 + C_2 \cos t + C_3 \sin t + \frac{1}{3}t^3 - 2t \)  
[20] \( y = C_1 + C_2e^{-4t} + \left( \frac{1}{3} \cos 2t - \sin 2t \right) \)  
[21] \( \{1, t, t^2, e^{2t}, e^{-2t}\} \)  
[22] (a) \( \frac{2s - 6}{s^2 - 2s} \) (b) \( \frac{12000}{s^6} \) (c) \( \frac{s}{s^2 - \pi^2} \) (d) \( \frac{60}{s^2 - 5}\)  
[23] (a) \( 3(e^{2t} - e^{-t}) \) (b) \( e^t + te^t \) (c) \( \frac{3}{3}e^t \) (d) \( 3e^{-t} \)  
[24] (a) \( y = \frac{1}{2}(e^{3t} + 4e^{-2t}) \) (b) \( y = \frac{1}{5}(\cos t - 2 \sin t + 4e^t \cos t - 2e^t \sin t) \)  
(c) \( y = -1 + \frac{1}{2}(e^t + e^{-t}) + u_5(t)(1 - \frac{1}{2}(e^{(t-5)} + e^{(-t-5)})) \),  
or \( y = -1 + \cos t + u_5(t)(-1 + \cos(t - 5)) \)  
(d) \( y = (-\frac{1}{8} \sin 2t + \frac{1}{4}) - u(t)(\frac{1}{8} \sin 2(t - 1) + \frac{1}{4}) - u(t)(\frac{1}{4} - \frac{1}{4} \cos 2(t - 1)) \)  
(e) \( y = 3(1 - e^{-t}) - 3u_2(t)(1 - e^{-t-2}) + u_4(t)(1 - e^{-t-4}) \)  
(f) \( y = \frac{1}{2}u_3(t)(t) \sin 2(t - 3) \)  
[25] \( \frac{100s}{(s + 2)(s^2 + \pi^2)} \)  
[26] \( \int_0^t (t - \tau) e^{3(t - \tau)}g(\tau) d\tau \) or \( \int_0^t \tau e^{3\tau}g(t - \tau) d\tau \)  
[27] \( x_1(t) = C_1e^{3t} + C_2e^{-t}, \ x_2(t) = 2C_1e^{3t} - 2C_2e^{-t} \)  
[28] Let \( x_1 = y, x_2 = y' \), then \( x_1' = x_2, x_2' = -3tx_1 - 2x_2 + \cos t \) , where \( x_1(0) = 1, x_2(0) = 4 \)  
[29] (a) \( \lambda_1 = 3, \ \nu^{(1)} = \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \); \( \lambda_2 = -1, \ \nu^{(2)} = \left( \begin{array}{c} 1 \\ -2 \end{array} \right) \)  
[29] (b) \( \lambda_1 = -1, \ \nu^{(1)} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \); \( \lambda_2 = -2, \ \nu^{(2)} = \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \)  
[30] \( x(t) = 2e^{3t} \left( \begin{array}{c} 1 \\ 2 \end{array} \right) + e^{-t} \left( \begin{array}{c} 1 \\ -2 \end{array} \right), \ \Phi(t) = \left( \begin{array}{c} e^{3t} \\ 2e^{3t} \\ -2e^{-t} \end{array} \right) \)  
[31] \( x(t) = 2e^{t} \left( \begin{array}{c} \sin t \\ \cos t \end{array} \right) - e^{-t} \left( \begin{array}{c} \cos t \\ -\sin t \end{array} \right) \)  
[32] \( x(t) = C_1e^{t} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + C_2 \left( \begin{array}{c} e^{t} \\ 0 \end{array} \right) + te^{t} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \)  
[33] \( \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)' = \left( \begin{array}{c} -\frac{1}{10} \\ \frac{9}{10} \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right), \ \left( \begin{array}{c} x_1(0) \\ x_2(0) \end{array} \right) = \left( \begin{array}{c} 50 \\ 75 \end{array} \right) \)  
Solution: \( \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = 50e^{-\frac{1}{10}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + 25e^{-\frac{1}{5}} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \)
\[
\begin{align*}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{10} & -\frac{1}{5} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 75 \end{pmatrix} \\
\text{Solution:} & \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{125}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{100}{3} e^{-\frac{2t}{10}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\begin{pmatrix} x(t) \end{pmatrix} &= C_1 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} x_p(t) \end{pmatrix} &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
\begin{pmatrix} x(t) \end{pmatrix} &= C_1 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\text{(i)} & \quad C \quad \text{(ii)} & \quad A \quad \text{(iii)} & \quad B \quad \text{(iv)} & \quad D
\end{align*}
\]
\[ f(t) = \mathcal{L}^{-1}\{F(s)\} \]
\[ F(s) = \mathcal{L}\{f(t)\} \]

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<table>
<thead>
<tr>
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</table>
| 1. | 1 | \[
\frac{1}{s}
\]
| 2. | \(e^{at}\) | \[
\frac{1}{s-a}
\]
| 3. | \(t^n\) | \[
\frac{n!}{s^{n+1}}
\]
| 4. | \(t^p\) \((p > -1)\) | \[
\frac{\Gamma(p + 1)}{s^{p+1}}
\]
| 5. | \(\sin at\) | \[
\frac{a}{s^2 + a^2}
\]
| 6. | \(\cos at\) | \[
\frac{s}{s^2 + a^2}
\]
| 7. | \(\sinh at\) | \[
\frac{a}{s^2 - a^2}
\]
| 8. | \(\cosh at\) | \[
\frac{s}{s^2 - a^2}
\]
| 9. | \(e^{at}\sin bt\) | \[
\frac{b}{(s-a)^2 + b^2}
\]
| 10. | \(e^{at}\cos bt\) | \[
\frac{s-a}{(s-a)^2 + b^2}
\]
| 11. | \(t^n e^{at}\) | \[
\frac{n!}{(s-a)^{n+1}}
\]
| 12. | \(u_c(t)\) | \[
\frac{e^{-cs}}{s}
\]
| 13. | \(u_c(t)f(t-c)\) | \[
e^{-cs}F(s)\]
| 14. | \(e^{at}f(t)\) | \[
F(s-c)
\]
| 15. | \(f(ct)\) | \[
\frac{1}{c} F\left(\frac{s}{c}\right), \ c > 0\]
| 16. | \[\int_0^t f(t-\tau)g(\tau)\,d\tau\] | \[
F(s)G(s)
\]
| 17. | \(\delta(t-c)\) | \[
e^{-cs}\]
| 18. | \(f^{(n)}(t)\) | \[
\sum_{k=0}^{n-1} s^k f^{(k)}(0) + \cdots + s f^{(n-2)}(0) - f^{(n-1)}(0)
\]
| 19. | \((-t)^n f(t)\) | \[
F^{(n)}(s)
\]