(1) **First Order Differential Equations.** (Separable, 1st Order Linear, Homogeneous, Exact)

(2) **Second Order Linear Homogeneous with Equations Constant Coefficients.**

The differential equation \( ay'' + by' + cy = 0 \) has **Characteristic Equation** \( ar^2 + br + c = 0 \).

Call the roots \( r_1 \) and \( r_2 \). The general solution of \( ay'' + by' + cy = 0 \) is as follows:

(a) If \( r_1, r_2 \) are real and distinct \( \Rightarrow \ y = C_1 e^{rt_1} + C_2 e^{rt_2} \)

(b) If \( r_1 = \lambda + i\mu \) (hence \( r_2 = \lambda - i\mu \)) \( \Rightarrow \ y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t \)

(c) If \( r_1 = r_2 \) (repeated roots) \( \Rightarrow \ y = C_1 e^{rt_1} + C_2 te^{rt_2} \)

(3) **Theory of 2nd Linear Order Equations.**

Wronskian of \( y_1, y_2 \) is \( W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \).

(a) The functions \( y_1(t) \) and \( y_2(t) \) are linearly independent over \( a < t < b \) if \( W(y_1, y_2) \neq 0 \) for at least one point in the interval.

(b) **THEOREM (Existence & Uniqueness)** If \( p(t), q(t) \) and \( g(t) \) are continuous in an open interval \( \alpha < t < \beta \) containing \( t_0 \), then the IVP \( \begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases} \)

has a unique solution \( y = \phi(t) \) defined in the open interval \( \alpha < t < \beta \).

(c) **Superposition Principle** If \( y_1(t) \) and \( y_2(t) \) are solutions of the 2nd order linear homogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = 0 \) over the interval \( a < t < b \), then \( y = C_1 y_1(t) + C_2 y_2(t) \) is also a solution for any constants \( C_1 \) and \( C_2 \).

(d) **THEOREM (Homogeneous)** If \( y_1(t) \) and \( y_2(t) \) are solutions of the linear homogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = 0 \) in some interval \( I \) and \( W(y_1, y_2) \neq 0 \) for some \( t_1 \) in \( I \), then the general solution is \( y_c(t) = C_1 y_1(t) + C_2 y_2(t) \). This is usually called the **complementary solution** and we say that \( y_1(t), y_2(t) \) form a **Fundamental Set of Solutions (FSS)** to the differential equation.

(e) **THEOREM (Nonhomogeneous)** The general solution of the nonhomogeneous equation

\( P(t)y'' + Q(t)y' + R(t)y = G(t) \)

is \( y(t) = y_c(t) + y_p(t) \), where \( y_c(t) = C_1 y_1(t) + C_2 y_2(t) \) is the general solution of the corresponding homogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = 0 \) and \( y_p(t) \) is a particular solution of the nonhomogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = G(t) \).

(f) **Useful Remark:** If \( y_{p_1}(t) \) is a particular solution of \( P(t)y'' + Q(t)y' + R(t)y = G_1(t) \) and if \( y_{p_2}(t) \) is a particular solution of \( P(t)y'' + Q(t)y' + R(t)y = G_2(t) \), then

\( y_p(t) = y_{p_1}(t) + y_{p_2}(t) \)

is a particular solution of \( P(t)y'' + Q(t)y' + R(t)y = [G_1(t) + G_2(t)] \).
(4) **Reduction of Order.** If \( y_1(t) \) is one solution of \( P(t)y'' + Q(t)y' + R(t)y = 0 \), then a second solution may be obtained using the substitution \( y = y_1(t) v(t) \). This reduces the original 2nd order equation to a 1st equation using the substitution \( w = \frac{dv}{dt} \). Solve that first order equation for \( w \), then since \( w = \frac{dv}{dt} \), solve this 1st order equation to determine the function \( v \).

(5) **Finding A Particular Solution \( y_p(t) \) to Nonhomogeneous Equations.**

You can always use the method of Variation of Parameters to find a particular solution \( y_p(t) \) of the linear nonhomogeneous equation \( y'' + p(t)y' + q(t)y = g(t) \). Variation of Parameters may require integration techniques.

If the coefficients of the differential equation are constants rather than functions and if \( g(t) \) has a very special form (see table below), it is usually easier to use Undetermined Coefficients:

(a) **Undetermined Coefficients** - **IF** \( ay'' + by' + cy = g(t) \) **AND** \( g(t) \) is as below:

<table>
<thead>
<tr>
<th>( g(t) )</th>
<th>Form of ( y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_m(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0 )</td>
<td>( t^s { A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0 } )</td>
</tr>
<tr>
<td>( e^{\alpha t} P_m(t) )</td>
<td>( t^s { e^{\alpha t} (A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0) } )</td>
</tr>
<tr>
<td>( e^{\alpha t} P_m(t) \cos \beta t ) or ( e^{\alpha t} P_m(t) \sin \beta t )</td>
<td>( t^s { e^{\alpha t} [F_m(t) \cos \beta t + G_m(t) \sin \beta t] } )</td>
</tr>
</tbody>
</table>

where \( s = \) the smallest nonnegative integer (\( s = 0, 1 \) or 2) such that no term of \( y_p(t) \) is a solution of the corresponding homogeneous equation. In other words, no term of \( y_p(t) \) is a term of \( y_c(t) \). (\( F_m(t), G_m(t) \) are both polynomials of degree \( m \).)

(b) **Variation of Parameters** - If \( y_1(t) \) and \( y_2(t) \) are two independent solutions of the homogeneous equation \( y'' + p(t)y' + q(t)y = 0 \), then a particular solution \( y_p(t) \) of the nonhomogeneous equation

\[
y'' + p(t)y' + q(t)y = g(t) \quad (\ast)
\]

has the form

\[
y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)
\]

where

\[
u_1' = \begin{bmatrix} 0 & y_2 \\ g(t) & y_2' \\ y_1 & y_2' \\ y_1' & y_2' \end{bmatrix}, \quad u_2' = \begin{bmatrix} y_1 & 0 \\ y_1' & g(t) \\ y_1' & y_2' \end{bmatrix}.
\]

**Remember:** Coefficient of \( y'' \) in (\( \ast \)) must be “1” in order to use the above formulas.
\begin{align*}
\text{(6) Spring-Mass Systems} \quad & \begin{cases}
m u'' + \gamma u' + ku = F(t) \\
u(0) = u_0, \ u'(0) = u_1
\end{cases}
\end{align*}

\begin{itemize}
\item \(m\) = mass of object, \(\gamma\) = damping constant, \(k\) = spring constant, \(F(t)\) = external force
\item Weight \(w = mg\), \textbf{Hooke's Law: } \(F_s = kd\)
\end{itemize}

\begin{align*}
\text{equilibrium: } F_s &= F_g \\
k d &= mg
\end{align*}

\begin{itemize}
\item \textbf{I. Undamped Free Vibrations: } \(m u'' + k u = 0\) \quad \text{(Simple Harmonic Motion)}
\item Note that \(A \cos \omega_0 t + B \sin \omega_0 t = R \cos (\omega_0 t - \delta), \text{ where } R = \sqrt{A^2 + B^2} = \text{amplitude}, \omega_0 = \text{frequency}, \frac{2\pi}{\omega_0} = \text{period} \text{ and } \delta = \text{phase shift} \text{ determined by } \tan \delta = \frac{B}{A}.
\item \textbf{II. Damped Free Vibrations: } \(m u'' + \gamma u' + ku = 0\)
\item (i) \(\gamma^2 - 4km > 0\) \quad \text{(overdamped } \iff\text{ distinct real roots to CE}
\item (ii) \(\gamma^2 - 4km = 0\) \quad \text{(critically damped } \iff\text{ repeated roots to CE}
\item (iii) \(\gamma^2 - 4km < 0\) \quad \text{(underdamped } \iff\text{ complex roots to CE (motion is oscillatory)}
\item \textbf{III. Forced Vibrations }:\quad (F(t) = F_0 \cos \omega t \text{ or } F(t) = F_0 \sin \omega t, \text{ for example})
\item (i) \(m u'' + \gamma u' + ku = F(t)\) \quad \text{(Damped) } \text{In this case if you write the general solution as } u(t) = u_T(t) + u_\infty(t), \text{ then } u_T(t) = \text{Transient Solution} \text{ (i.e. the part of } u(t) \text{ such that } u_T(t) \to 0 \text{ as } t \to \infty) \text{ and } u_\infty(t) = \text{Steady-State Solution} \text{ (the solution behaves like this function in the long run).}
\item (ii) \(m u'' + ku = F_0 \cos \omega t\) \quad \text{(Undamped) } \text{If } \omega = \omega_0 = \sqrt{\frac{k}{m}} \Rightarrow \text{Resonance occurs and the solution is unbounded; while if } \omega \neq \omega_0 \text{ then motion is a series of beats} \text{ (solution is bounded) }
\end{itemize}

\begin{align*}
\text{(7) } n^{\text{th}} \text{ Order Linear Homogeneous Equations With Constant Coefficients} \quad & a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \quad (*)
\end{align*}

This differential equation has \(n\) independent solutions.

\textit{Characteristic Equation: } \(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0\) \text{ will have } \(n\) \text{ characteristic roots that may be real and distinct, repeated, complex, or complex and repeated.}
(a) For each real root \( r \) that is not repeated \( \Rightarrow \) get a solution of \((\ast)\): \( e^{rt} \)

(b) For each real root \( r \) that is repeated \( m \) times \( \Rightarrow \) get \( m \) independent solutions of \((\ast)\): \( e^{rt}, te^{rt}, t^2 e^{rt}, \ldots, t^{m-1} e^{rt} \)

(c) For each complex root \( r = \lambda + i\mu \) repeated \( m \) times \( \Rightarrow \) get \( 2m \) solutions of \((\ast)\): \( e^{\lambda t} \cos \mu t, te^{\lambda t} \cos \mu t, \ldots, t^{m-1} e^{\lambda t} \cos \mu t \) \( \text{and} \) \( e^{\lambda t} \sin \mu t, te^{\lambda t} \sin \mu t, \ldots, t^{m-1} e^{\lambda t} \sin \mu t \)

(don’t need to consider its conjugate root \( \lambda - i\mu \))

(8) Undetermined Coefficients for \( n^{th} \) Order Linear Equations

This can only be used to find \( y_p(t) \) of \( a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \) and \( g(t) \) one of the 3 very SPECIAL FORMS in table in (5) above. The particular solution has the same form as before: \( y_p(t) = t^s \left[ \cdots \right] \), where \( s = \) the smallest nonnegative integer such that no term of \( y_p(t) \) is a term of \( y_c(t) \), except this time \( s = 0, 1, 2, \ldots, n \).

(9) Laplace Transforms

(a) Be able to compute Laplace transforms using definition:

\[
\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt
\]

and using a table of Laplace transforms (see table on page 317) and using linearity:

\[
\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \quad \text{and} \quad \mathcal{L}\{cf(t)\} = c \mathcal{L}\{f(t)\}.
\]

(b) Computing Inverse Laplace Transforms: Must be able to use a table of Laplace transforms usually together with Partial Fractions or Completing the Square, to find inverse Laplace transforms: \( f(t) = \mathcal{L}^{-1}\{F(s)\} \).

(c) Solving Initial Value Problems: Recall that

\[
\mathcal{L}\{y'\} = s \mathcal{L}\{y\} - y(0)
\]

\[
\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - s y(0) - y'(0)
\]

\[
\mathcal{L}\{y'''\} = s^3 \mathcal{L}\{y\} - s^2 y(0) - s y'(0) - y''(0)
\]

\[\vdots\]

(d) Discontinuous Functions:

(i) **Unit Step Function** (Heaviside Function): If \( c \geq 0 \), \( u_c(t) = \begin{cases} 
0, & t < c \\
1, & t \geq c
\end{cases} \)

\[
\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}
\]
(ii) **Unit “Pulse” Function**:  \[ u_a(t) - u_b(t) = \begin{cases} 
1, & a \leq t < b \\
0, & \text{otherwise} 
\end{cases} \]

(iii) **Translated Functions**:  
\[ y = g(t) = \begin{cases} 
0, & t < c \\
f(t-c), & t \geq c 
\end{cases} = u_c(t)f(t-c). \]

\[ L\{u_c(t)f(t-c)\} = e^{-cs}F(s), \text{ where } F(s) = L\{f(t)\} \]

Thus,  
\[ L^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c), \text{ where } f(t) = L^{-1}\{F(s)\} \]

A useful formula **NOT** in the book:  
\[ L\{u_c(t)h(t)\} = e^{-cs}L\{h(t+c)\} \]

(iv) **Unit Impulse Functions**:  
\[ y = \delta(t-c) \ (c \geq 0), \text{ then } L\{\delta(t-c)\} = e^{-cs} \]

(e) **Convolutions**:  
\[ L\{(f * g)(t)\} = L\left\{ \int_0^t f(t-\tau) g(\tau) d\tau \right\} = L\{f(t)\} L\{g(t)\} \]

---

**Practice Problems**

[1] For what value of \( \alpha \) will the solution to the IVP  
\[ \begin{cases} 
y'' - y' - 2y = 0 \\
y(0) = \alpha \\
y'(0) = 2 
\end{cases} \]

satisfy \( y \to 0 \) as \( t \to \infty \)?

[2] (a) Show that \( y_1 = x \) and \( y_2 = x^{-1} \) are solutions of the differential equation \( x^2y'' + xy' - y = 0 \).

(b) Evaluate the Wronskian \( W(y_2, y_1) \) at \( x = \frac{1}{2} \).

(c) Find the solution of the initial value problem \( x^2y'' + xy' - y = 0, \ y(1) = 2, \ y'(1) = 4 \).
[3] Find the largest open interval for which the initial value problem
\[ 3x^2y'' + y' + \frac{1}{x - 2}y = \frac{1}{x - 3}, \quad y(1) = 3, \quad y'(1) = 2, \text{ has a solution.} \]

In Problems 4, 5, and 6 find the general solution of the homogeneous differential equations in (a) and use the method of Undetermined Coefficients to find a particular solution \( y_p \) in (b) and find the FORM of a particular solution (c).

[4] (a) \( y'' - 5y' + 6y = 0 \) (b) \( y'' - 5y' + 6y = t^2 \) (c) \( y'' - 5y' + 6y = e^{2t} + \cos(3t) \)

[5] (a) \( y'' - 6y' + 9y = 0 \) (b) \( y'' - 6y' + 9y = te^{3t} \) (c) \( y'' - 6y' + 9y = e^t + \cos(3t) \)

[6] (a) \( y'' - 2y' + 10y = 0 \) (b) \( y'' - 2y' + 10y = e^x + \cos(3x) \) (c) \( y'' - 2y' + 10y = e^x \cos(3x) \)

[7] Find the general solution to (a) \( y'' + y' - 6y = 7e^{4t} \) (b) \( y'' + y' - 6y = 7e^{4t} - 100 \sin t \)

[8] Solve this IVP: \( y'' - y' = 4t, \quad y(0) = 0, \quad y'(0) = 0. \)

[9] Find the general solution to \( y'' + y = \tan t, \quad 0 < x < \frac{\pi}{2}. \)

[10] The differential equation \( x^2y'' - 2xy' + 2y = 0 \) has solutions \( y_1(x) = x \) and \( y_2(x) = x^2 \). Use the method of Variation of Parameters to find a solution of \( x^2y'' - 2xy' + 2y = 2x^2 \).

[11] The differential equation \( x^2y'' + xy' - y = 0 \) has one solution \( y_1(x) = x \). Use the method of Reduction of Order to find a second (linearly independent) solution of \( x^2y'' + xy' - y = 0. \)

[12] For what nonnegative values of \( \gamma \) will the solution of the initial value problem 
\[ u'' + \gamma u' + 4u = 0, \quad u(0) = 4, \quad u'(0) = 0 \text{ oscillate?} \]

[13] (a) For what positive values of \( k \) does the solution of the initial value problem 
\[ 2u'' + ku = 3 \cos(2t), \quad u(0) = 0, \quad u'(0) = 0, \text{ become unbounded (Resonance)?} \]

[14] Find the steady–state solution of the IVP \( y'' + 4y' + 4y = \sin t, \quad y(0) = 0, \quad y'(0) = 0. \)

[15] A 4-kg mass stretches a spring 0.392 m. If the mass is released from 1 m below the equilibrium position with a downward velocity of 10 m/sec, what is the maximum displacement?

In Problems 16 and 17 find the general solution of the homogeneous differential equations in (a) and use the method of Undetermined Coefficients to find the FORM of a particular solution of the nonhomogeneous equation in (b).

[16] (a) \( y'' - y' = 0 \) (b) \( y'' - y' = t + e^t \)

[17] (a) \( y'' - y' - y + y = 0 \) (b) \( y'' - y' - y' + y = e^t + \cos t \)

[18] Find the solution of the initial value problem \( y'' - 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 1. \)

[19] Find the general solution of the differential equation \( y'' + y' = t^2. \)

[20] Find the general solution of \( y'' + 4y' = -10 \sin 2t. \)

[21] Find a fundamental set of solutions of \( y^{(5)} - 4y'' = 0. \)

[22] Find the Laplace transform of these functions:
(a) \( f(t) = 3 - e^{2t} \) (b) \( g(t) = 100t^5 \) (c) \( h(t) = \cosh \pi t \) (d) \( k(t) = -10t^3e^{5t} \)

[23] Find the inverse Laplace transform of
(a) \( F(s) = \frac{9}{s^2 - s - 2} \) (b) \( F(s) = \frac{s}{(s - 1)^2} \) (c) \( F(s) = \frac{8}{(s + 1)^4} \) (d) \( F(s) = \frac{3s + 2}{s^2 + 2s + 5} \)

[24] Solve these initial value problems:
(a) \( \begin{cases} y'' - y' - 6y = 0 \\ y(0) = 1 \\ y'(0) = -1 \end{cases} \)

(b) \( \begin{cases} y'' - 2y' + 2y = \cos t \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \)

(c) \( y'' - y = \begin{cases} 1, & t < 5 \\ 2, & 5 \leq t < \infty \end{cases}; \quad y(0) = y'(0) = 0. \)
(d) \( y'' + 4y = \begin{cases} t, & t < 1 \\ 0, & 1 < t < \infty \end{cases} \); \( y(0) = y'(0) = 0 \).

(e) \( y' + g(t), \ y(0) = 0 \) and where \( g(t) \):

\[
\begin{array}{c}
y = g(t) \\
0 \quad 2 \quad 4
\end{array}
\]

(f) \( y'' + 4y = \delta(t - 3), \ y(0) = y'(0) = 0 \)

[25] \( \mathcal{L} \left\{ \int_0^t 100 e^{-2\tau} \cos \pi(t - \tau) \, d\tau \right\} = ? \)

[26] If \( g(t) = \mathcal{L}^{-1}\{G(s)\} \), then \( \mathcal{L}^{-1} \left\{ \frac{G(s)}{(s - 3)^2} \right\} = ? \)

**Answers**

1. \( \alpha = -2 \)  
2. (b) \( W(x^{-1}, x)\left(\frac{1}{2}\right) = 4 \); (c) \( y = 3x - x^{-1} \)  
3. \( 0 < x < 2 \)

4. (a) \( y = C_1 e^{2t} + C_2 e^{3t} \)  
   (b) \( y = At^2 + Bt + C \)  
   (c) \( y = Ate^{2t} + B \cos(3t) + C \sin(3t) \)

5. (a) \( y = C_1 e^{3t} + C_2 e^{2t} \)  
   (b) \( y = Ate^{2t} + B \cos(3t) + C \sin(3t) \)

6. (a) \( y = C_1 e^{x} \cos(3x) + C_2 e^{x} \sin(3x) \)  
   (b) \( y = A e^{2x} + B \cos(3x) + C \sin(3x) \)  
   (c) \( y = x(A \cos(3x) + B \sin(3x)) e^{x} \)

7. (a) \( y = C_1 e^{-3t} + C_2 e^{2t} + \frac{1}{2} e^4t \)  
   (b) \( y = Ate^{-3t} + C_2 e^{2t} + \frac{1}{2} e^4t + 2 \cos t + 14 \sin t \)

8. \( y = -4 + 4e^t - 2t^2 - 4t \)

9. \( y = C_1 \cos t + C_2 \sin t - (\cos t) \ln(\sec t + \tan t) \)

10. \( y = 2x^2 \ln x \) or \( y = 2x^2 \ln x + (C_1 x + C_2 x^2) \)

11. \( y = x - 1 \) or \( y = Ax^{-1} + Bx, \ A \neq 0 \)

12. \( 0 \leq \gamma < 4 \)

13. (a) \( k = 8 \) (resonance)  
   (b) NO value of \( k \), all solutions are bounded.

14. \( y = \frac{1}{25} (3 \sin t - 4 \cos t) \)

15. \( u(t) = \cos 5t + 2 \sin 5t = \sqrt{5} \cos (5t - \delta) \), \( \delta = \tan^{-1} 2 \approx 1.1 \) Thus amplitude = \( \sqrt{5} \).

16. (a) \( y = C_1 + C_2 e^{-t} + C_3 e^{t} \)  
   (b) \( y = t(At + B) + Cte^t \)

17. (a) \( y = C_1 e^{t} + C_2 e^{2t} + C_3 e^{-t} \)  
   (b) \( y = At^2 e^t + B \cos t + C \sin t \)

18. \( y = 3 - e^t + te^t \)

19. \( y = C_1 + C_2 \cos t + C_3 \sin t + \frac{1}{3} t^3 - 2t \)

20. \( y = C_1 + C_2 e^{-4t} + \left(\frac{1}{2}\right) \cos 2t - \sin 2t \)

21. \( \{t, t^2, e^{2t}, e^{-2t}\} \)

22. (a) \( \frac{2s - 6}{s^2 - 2s} \)  
   (b) \( \frac{1200}{s^6} \)  
   (c) \( \frac{s}{s^2 - \pi^2} \)  
   (d) \( -\frac{60}{(s - 5)^4} \)

23. (a) \( 2(\frac{e^{2t} - e^{-t}}{2}) \)  
   (b) \( e^t + te^t \)  
   (c) \( \frac{3t^3 e^{-t}}{4} \)  
   (d) \( 3e^{-t} \cos 2t - \frac{1}{2} e^{-t} \sin 2t \)

24. (a) \( y = \frac{1}{2} (e^{3t} + 4e^{-2t}) \)  
   (b) \( y = \frac{1}{2} (\cos t - 2 \sin t + 4e^t \cos t - 2e^t \sin t) \)
   (c) \( y = -1 + \frac{1}{2} (e^t + e^{-t}) + u_3(t)(-1 + \frac{1}{2} (e^{(t-5)} + e^{-(t-5)})), \)
   \( \text{or } y = -1 + \cosh t + u_3(t)(-1 + \cosh(t - 5)) \)
   (d) \( y = \left(-\frac{1}{8} \sin 2t + \frac{1}{3}\right) - u_4(t)(u_4(t)(-1 + \sin(2(t - 1)) + u_4(t)(\frac{1}{4} - \frac{1}{2} \cos 2(t - 1)) \))
   (e) \( y = 3(1 - e^{-t}) - 3u_2(t)(1 - e^{-(t-2)}) + 3u_4(t)(1 - e^{-(t-4)}) \)
   (f) \( y = \frac{1}{2} u_3(t)(t) \sin 2(t - 3) \)

25. \( \frac{100 s}{(s + 2)(s^2 + \pi^2)} \)

26. \( \int_0^t (t - \tau) e^{3(t-\tau)} g(\tau) \, d\tau \) or \( \int_0^t \tau e^{3\tau} g(t - \tau) \, d\tau \)