SOLUTIONS

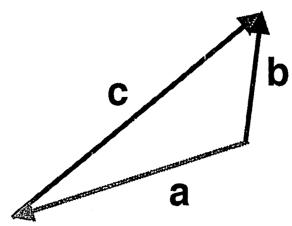
MA 362 - Fall 2016

Homework Set # 1

1. (§1.1) **Page 18:** # 13, 16, 21, 37.

2. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, prove the following:
   (a) *Parallelogram Law* : $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$
   (b) *Polarization Identity* : $||\mathbf{u}+\mathbf{v}||^2 - ||\mathbf{u}-\mathbf{v}||^2 = 4(\mathbf{u}\cdot\mathbf{v})$

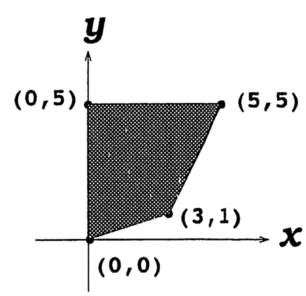
3. Prove the converse of the Pythagorean Theorem as follows:
   Suppose the sides of a triangle are the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (as shown below). If $||\mathbf{a}||^2 + ||\mathbf{b}||^2 = ||\mathbf{c}||^2$, prove that $\mathbf{a}\cdot\mathbf{b} = 0$ (i.e., the triangle must be a right triangle).



4. (§1.2) **Page 29:** # 20, 26.

5. (§1.3) **Page 49:** # 4, 6, 11, 29, 33, 34.

6. Find the area of the polygon Ω shown here:



7. The line through the points $(3, 2, 1)$ and $(5, 1, 2)$ intersects the plane $x + y + z = 14$ at what point?

8. The plane containing the points $P(1, 1, 1)$, $Q(2, 0, -4)$, and $R(1, 2, 3)$ intersects the $x-$axis at what point?

9. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Compute A^2 and A^3. Use mathematical induction to prove that $A^n = 3^{n-1}A$ for any positive integer n.

1 page 18 #13: $\vec{v}_1 = \langle 2,7,0\rangle$, $\vec{v}_2 = \langle 0,2,7\rangle$

Soln 1: Plane spanned by $\vec{v}_1, \vec{v}_2$ is set of all linear combinations of $\vec{v}_1, \vec{v}_2$: $a\vec{v}_1 + b\vec{v}_2 \quad (a,b \in \mathbb{R})$

Hence plane is $\{(2a, 7a+2b, 7b) : a,b \in \mathbb{R}\}$ ✓

(Note that $(0,0,0)$ belongs to this plane)

The form above is the "vector form" of a plane – we'll see more like this when we study surface integrals.

Soln 2: Find equation of plane through $(0,0,0)$ and containing $\vec{v}_1$ and $\vec{v}_2$. Clearly a normal vector to plane is $\vec{N} = \vec{v}_1 \times \vec{v}_2 = \langle 49, -14, 4\rangle$. Hence eqn of plane is $\langle x-x_0, y-y_0, z-z_0\rangle \cdot \vec{N} = 0$, where $(x_0,y_0,z_0) = (0,0,0)$.

$\therefore 49x - 14y + 4z = 0$

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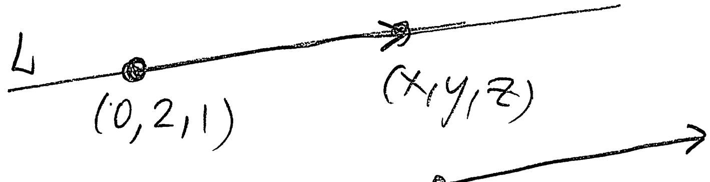
page 18 #16: want eqn of line through $(0,2,1)$ in the direction $\vec{d} = 2\vec{i} - \vec{k}$:



$\Rightarrow \langle x-0, y-2, z-1\rangle = t\vec{d}$

$\vec{d} = \langle 2,0,-1\rangle$

or $\langle x,y,z\rangle = \langle 0,2,1\rangle + t\langle 2,0,-1\rangle$ ← Vector Form of L

or $\begin{cases} x = 2t \\ y = 2 \\ z = 1-t \end{cases}$ ← Parametric Form of L

page 18 #21: $P(2,3,-4)$, $Q(2,1,-1)$, $R(2,7,-10)$
~~Do~~ these lie on same line?

Soln 1: Eqn of line through P, Q is

$L: \vec{r}(t) = \langle 2,3,-4\rangle + t\,\overrightarrow{PQ} = \langle 2,3,-4\rangle + t\langle 0,-2,3\rangle$

i.e. $\begin{cases} x = 2 \\ y = 3-2t \\ z = -4+3t \end{cases}$ Now $R(2,7,-10)$ lies on $L \iff$

$\begin{cases} 2 = 2 \\ 7 = 3-2t \\ -10 = -4+3t \end{cases}$ is true for some value of t

$t=-2$ works.

$\therefore$ YES P, Q, R are collinear

Soln 2: P, Q, R are collinear if $\overrightarrow{PQ} \parallel \overrightarrow{PR}$

Now $\overrightarrow{PQ} = \langle 0,-2,3\rangle$ and $\overrightarrow{PR} = \langle 0,4,-6\rangle$

Since $-2\,\overrightarrow{PQ} = \overrightarrow{PR} \Rightarrow \overrightarrow{PQ}$ is parallel to $\overrightarrow{PR}$

Hence $\vec{P}, \vec{Q}, \vec{R}$ must lie on same line

③

page 18: # 37 : Find line which lies entirely on the quadric surface $x^2+y^2-z^2=1$ ← hyperboloid of 1-sheet

Examples are easy to find. Here is how you can find all of them:

Every line has form $\begin{cases} x = a+bt \\ y = c+dt \\ z = e+ft \end{cases}$

Now if line lies on $x^2+y^2-z^2=1$ then

$$(a+bt)^2+(c+dt)^2-(e+ft)^2=1$$

$$(a^2+c^2-e^2)+2(ab+cd+ef)t+(b^2+d^2-f^2)t^2=1$$

Thus the only conditions on the parameters are

$$\begin{cases} a^2+c^2-e^2=1 \\ ab+cd+ef=0 \\ b^2+d^2-f^2=0 \end{cases}$$

Hence an easy example is $a=0,\ b=1,\ c=1,\ d=0,\ e=0,\ f=1$

$L: \begin{cases} x=t \\ y=1 \\ z=t \end{cases}$ Here is a harder example: $\begin{cases} x=2+t \\ y=t/\sqrt{3} \\ z=\sqrt{3}+\frac{2}{\sqrt{3}}t \end{cases}$

2 $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$

$= \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$ ✓

$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$

$= \|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$ ✓

Hence $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$ and

$\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = 4(\vec{u}\cdot\vec{v})$

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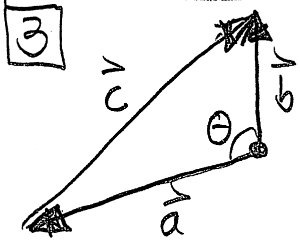

Given that $\|\vec{a}\|^2 + \|\vec{b}\|^2 = \|\vec{c}\|^2$ and from the vector diagram, $\vec{c} = -\vec{a} + \vec{b}$

$\therefore \|\vec{a}\|^2 + \|\vec{b}\|^2 = \|\vec{c}\|^2 = \|-\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 - 2(\vec{a}\cdot\vec{b}) + \|\vec{b}\|^2$

$\Rightarrow 0 = -2(\vec{a}\cdot\vec{b}) \Rightarrow \vec{a} \perp \vec{b}$ (i.e. $\theta = \frac{\pi}{2}$)

Or: You can use Law of Cosines: $\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$

~~So~~ if $\|\vec{a}\|^2 + \|\vec{b}\|^2 = \|\vec{c}\|^2 \Rightarrow -2\|\vec{a}\|\,\|\vec{b}\|\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ ∎

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4 <u>page 29 #20</u>: $\vec{u} = \langle -1, 1, 1\rangle$, $\vec{v} = \langle 2, 1, -3\rangle$

$\therefore \operatorname{proj}_{\vec{v}} \vec{u} = \left(\dfrac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\dfrac{-2+1-3}{14}\right)\vec{v} = -\dfrac{2}{7}\vec{v}$ ✓

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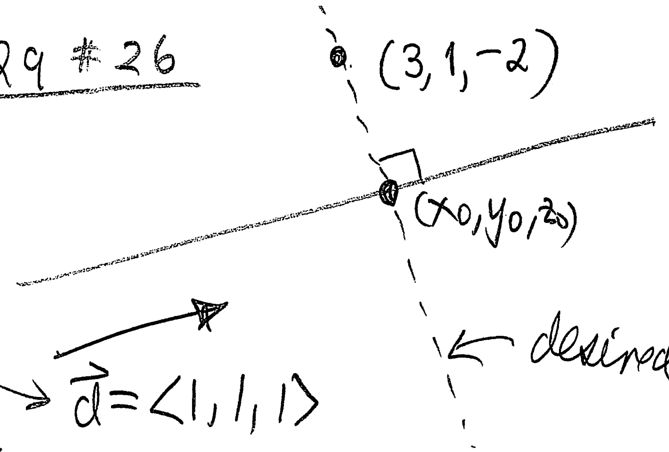
page 29 # 26



$$L: \begin{cases} x = -1+t \\ y = -2+t \\ z = -1+t \end{cases}$$

direction vector for L $\rightarrow$ $\vec{d} = \langle 1,1,1\rangle$

$\leftarrow$ desired line L^*

Since $L^* \perp L \Rightarrow \langle x_0-3, y_0-1, z_0+2\rangle \cdot \langle 1,1,1\rangle = 0$

$\Rightarrow x_0+y_0+z_0 = 2$ and since $(x_0,y_0,z_0) \in L$

$\Rightarrow (-1+t)+(-2+t)+(-1+t) = 2 \Rightarrow t=2$

So $(x_0,y_0,z_0) = (1,0,1)$. Now just find eqn of line L^* through $P(3,1,-2)$ and $Q(1,0,1)$:

$\vec{r}(t) = \langle 3,1,-2\rangle + t\overrightarrow{PQ} = \langle 3,1,-2\rangle + t\langle -2,-1,3\rangle$ ✓

or $\begin{cases} x = 3-2t \\ y = 1-t \\ z = -2+3t \end{cases}$ ✓

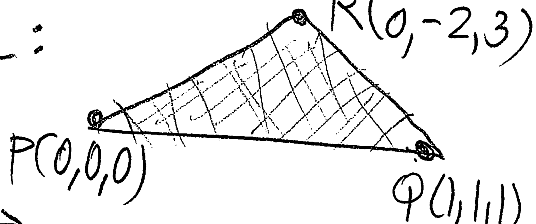
$\boxed{5}$ page 49 # 4: $\vec{a} = \langle 1,-2,1\rangle$, $\vec{b} = \langle 2,1,1\rangle$, $\vec{c} = \langle 3,-1,2\rangle$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \langle 3, -(1), -5\rangle = \langle 3,-1,-5\rangle$$

$\therefore \vec{a}\cdot(\vec{b}\times\vec{c}) = \langle 1,-2,1\rangle \cdot \langle 3,-1,-5\rangle = 3+2-5 = 0$ ✓

Note: Geometrically, this says that $\vec{a},\vec{b},\vec{c}$ are coplanar (they lie on same plane).

page 49 #6:



$$\text{Area} = \frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\|\langle 1,1,1\rangle \times \langle 0,-2,3\rangle\|$$
$$= \frac{1}{2}\|\langle 5,-3,-2\rangle\| = \frac{1}{2}\sqrt{38} \checkmark$$

Note: Also $\text{Area} = \frac{1}{2}\|\vec{QR} \times \vec{QP}\| = \frac{1}{2}\|\langle -1,-3,2\rangle \times \langle -1,-1,-1\rangle\|$
$$= \frac{1}{2}\sqrt{38}$$

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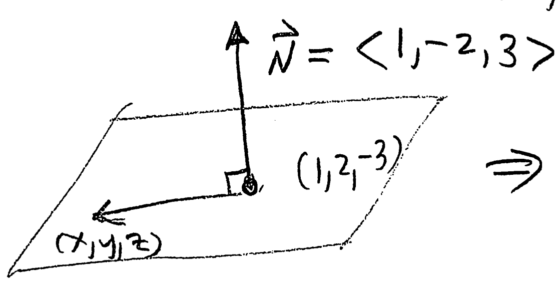
page 49 #11: $\vec{u} = \langle -5, 9, -4\rangle$, $\vec{v} = \langle 7, 8, 9\rangle$

There are only two unit vectors $\perp$ to both $\vec{u}$ and $\vec{v}$:

$$\pm \frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|} = \pm \frac{\langle 113, 17, -103\rangle}{\sqrt{(113)^2 + (17)^2 + (103)^2}} \checkmark$$

Page 49 # 29: Eqn of plane through $(1,2,-3)$ and $\perp$ line $\vec{v} = \langle 0,-2,1\rangle + t\langle 1,-2,3\rangle$



$$\Rightarrow \langle x-x_0, y-y_0, z-z_0\rangle \cdot \vec{N} = 0$$
$$\langle x-1, y-2, z+3\rangle \cdot \langle 1,-2,3\rangle = 0$$
$$\Rightarrow x - 2y + 3z + 12 = 0 \checkmark$$

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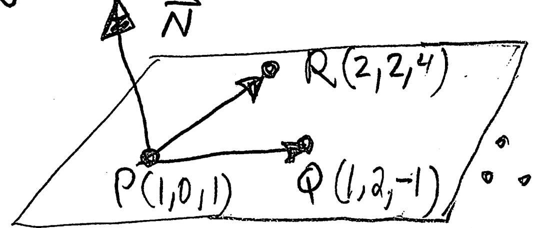
page 49 #33: Eqn of plane containing $(1,0,1)$ and line $\vec{l}(t) = \langle 1,2,-1\rangle + t\langle 1,0,5\rangle$. We just need 3 non-collinear points to uniquely determine the plane. $P(1,0,1)$ is given, so choose 2 points on line, say $Q(1,2,-1)$ and $R(2,2,4)$. These occur when $t=0$ and $t=1$.



$\vec{N} = \vec{PQ} \times \vec{PR} = \langle 10,-2,-2\rangle$

$\therefore$ Eqn of plane is

$$\langle x-x_0, y-y_0, z-z_0\rangle \cdot \vec{N} = 0$$

$$\langle x-1, y-0, z-1\rangle \cdot \langle 10,-2,-2\rangle = 0$$

$$\Rightarrow \quad 5x-y-z=4 \quad ✓$$

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page 49 # 34: Distance from (x_1,y_1,z_1) to $Ax+By+Cz+D=0$ is

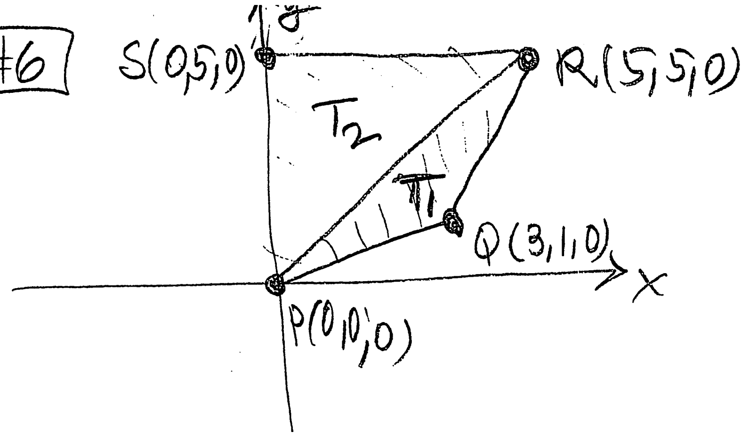
$$\text{distance} = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

point is $(2,1,-1)$

plane is $x-2y+2z+5=0$

$$\therefore \text{distance} = \frac{|1(2)-2(1)+2(-1)+5|}{\sqrt{1^2+(-2)^2+(2)^2}} = \frac{3}{3} = 1 \quad ✓$$

$\boxed{\#6}$



Area $T_1 = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\|\langle 0,0,10\rangle\| = 5$

Area $T_2 = \frac{1}{2}\|\overrightarrow{PR} \times \overrightarrow{PS}\| = \frac{1}{2}\|\langle 0,0,25\rangle\| = \frac{25}{2}$

$\therefore$ Area $\Omega = 5 + \frac{25}{2} = \frac{35}{2}$ ✓

$\boxed{\#7}$ Line through $P(3,2,1)$, $Q(5,1,2)$ is

$$\vec{r}(t) = \langle 3,2,1\rangle + t\,\overrightarrow{PQ} = \langle 3,2,1\rangle + t\langle 2,-1,1\rangle$$

In parametric form: $L: \begin{cases} x = 3+2t \\ y = 2-t \\ z = 1+t \end{cases}$

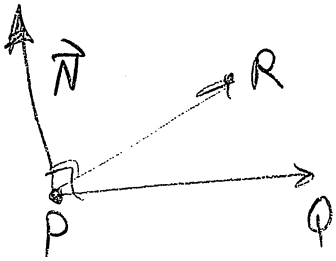
L intersects the plane $x+y+z=14$

when $(3+2t)+(2-t)+(1+t)=14$

i.e. $t=4$

Hence the intersection point is $(11,-2,5)$ ✓

#8 Plane containing $P(1,1,1)$, $Q(2,0,-4)$, $R(1,2,3)$



has normal vector $\vec{N} = \vec{PQ} \times \vec{PR} = \langle 3,-2,1 \rangle$

So eqn of plane is $\langle x-1, y-1, z-1 \rangle \cdot \langle 3,-2,1 \rangle = 0$

or $3x - 2y + z = 2$. Plane crosses x-axis when $y = z = 0$

Thus $x = 2/3$, so point is $(2/3, 0, 0)$ ✓

#9 $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 3A$

and $A^3 = (A^2)A = \{3A\}A = 3A^2 = 3(3A) = 3^2 A$

Show $\underline{A^n = 3^{n-1} A}$ (*) for any $n = 1, 2, 3, \ldots$

Clearly (*) is true for $n = 1$ since $A^1 = 3^0 A = A$ ✓

Assume (*) is true for $n = N$. It remains to show that this implies (*) is true for $n = N+1$:

note that $A^{N+1} = A^N A \underset{\substack{\uparrow \\ \text{Induction hypothesis}}}{=} (3^{N-1}A)A = 3^{N-1}A^2 = 3^{N-1}(3A)$

$= 3^N A$

Hence Induction hypothesis implies (*) true for $n = N+1$.

Thus (*) holds for all $n \in \mathbb{N}$ ■