SOLUTIONS

MA 362 - Fall 2016

Homework Set # 3

1. (§2.2) **Page 103:** # 8, 12(a)(b).

2. If $f(x,y) = (2x,\, 3\cos y,\, \ln x)$ and $g(x,y) = (x^2+y^2,\, -2xy)$, then find $f \circ g$ and $g \circ f$, if possible. Compute $f(g(1,0))$ and $g(f(1,0))$, if possible.

3. (§2.3) **Page 115:** # 3(a)(c), 5, 10(a)(c), 11, 12, 16(a).

4. If $w = \dfrac{r^2+s^2}{r^2-s^2}$, show that $r\,\dfrac{\partial w}{\partial r} + s\,\dfrac{\partial w}{\partial s} = 0$, for all $r \neq s$.

5. Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is the *linear* mapping $f(\mathbf{x}) = A\mathbf{x}$, where A is the 3×2 matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$.

   Compute the matrix of partials Df.

   Note: $f(x_1,\, x_2) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and write RHS as a row.

1 page 103 #8

(a) $\lim\limits_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy} = \lim\limits_{(x,y)\to(0,0)} \frac{4xy}{xy} = \lim\limits_{(x,y)\to(0,0)} 4 = 4$ ✓

(b) Soln 1: $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{y} = \lim\limits_{(x,y)\to(0,0)} x\frac{\sin(xy)}{(xy)}$

$= \left(\lim\limits_{(x,y)\to(0,0)} x\right)\left(\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{(xy)}\right)$ (since both limits exist)

$= 0\cdot 1 = 0$ ✓ $\left[\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{(xy)} = \lim\limits_{r\to 0} \frac{\sin r}{r} = 1\right]$

$r = xy$

Soln 2: $0 \le \left|\frac{\sin(xy)}{y}\right| = \left|x\frac{\sin(xy)}{xy}\right| \le |x|\left|\frac{\sin(xy)}{xy}\right| \le C|x|$

(since $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{(xy)} = 1 \Rightarrow \frac{\sin(xy)}{xy}$ is bounded.)

Now use Squeeze Thm!

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$ ✓

let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Page 103 #12

(a) $\lim\limits_{x\to 0} \frac{\sin 2x - 2x}{x^3} = \lim\limits_{x\to 0} \frac{2\cos 2x - 2}{3x^2} = \lim\limits_{x\to 0} \frac{-4\sin 2x}{6x} = \lim\limits_{x\to 0} \frac{-8\cos 2x}{6} = -\frac{4}{3}$ ✓

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin 2x - 2x + y}{x^3+y}$ <u>DNE</u> since

along $y=0$, previous problem $\Rightarrow \frac{\sin 2x - 2x}{x^3} \to -\frac{4}{3}$

but, along $x=0$, we get $\frac{y}{y} \to 1$

$\boxed{2}$ $f(x,y) = (2x,\ 3\cos y,\ \ln x)$; $g(x,y) = (x^2+y^2,\ -2xy)$

Since
$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[rr, bend right] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}^3 \end{tikzcd}$$

$f \circ g$ is defined, but $g \circ f$ <u>NOT</u> defined

$$(f\circ g)(x,y) = f(g(x,y)) = f\left(\{x^2+y^2\}, \{-2xy\}\right)$$
$$= \left(2\{x^2+y^2\},\ 3\cos\{-2xy\},\ \ln\{x^2+y^2\}\right) \checkmark$$

and $f(g(1,0)) = (2, 3, 0)$ $\checkmark$

$\boxed{3}$ <u>page 115 #3:</u>

(a) $\underline{w = x e^{(x^2+y^2)}} \Rightarrow \dfrac{\partial w}{\partial x} = (2x^2+1)e^{(x^2+y^2)}$, $\dfrac{\partial w}{\partial y} = 2xy\, e^{(x^2+y^2)}$

(c) $\underline{w = e^{xy} \ln(x^2+y^2)}$

$$\Rightarrow \frac{\partial w}{\partial x} = e^{xy}\left(\frac{2x}{x^2+y^2}\right) + y e^{xy} \ln(x^2+y^2)$$

$$\frac{\partial w}{\partial y} = e^{xy}\left(\frac{2y}{x^2+y^2}\right) + x e^{xy} \ln(x^2+y^2)$$

page 115 #5: $z = f(x,y) = x^2 + y^3$ $(x_0, y_0, z_0) = (3,1,10)$

Eqn of tangent plane $z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

$f_x = 2x, \; f_y = 3y^2$

$\therefore \; z = 10 + 6(x-3) + 3(y-1)$ ✓

page 115 # 10:

ⓐ $f(x,y) = (\overset{f_1}{e^x}, \overset{f_2}{\sin xy})$. Since $f: \mathbb{R}^2 \to \mathbb{R}^2$, Df is 2×2 matrix

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^x & 0 \\ y\cos xy & x\cos xy \end{bmatrix}$$ ✓

ⓒ $f(x,y) = (\overset{f_1}{x+y}, \overset{f_2}{x-y}, \overset{f_3}{xy})$. Since $f: \mathbb{R}^2 \to \mathbb{R}^3$, Df is 3×2 matrix

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ y & x \end{bmatrix}$$ ✓

page 115 #11: $z = f(x,y) = x^2 - 2xy + 2y^2$

Given: $\frac{\partial f}{\partial x}(x_0,y_0) = 2$ and $\frac{\partial f}{\partial y}(x_0,y_0) = 4$

Since $\frac{\partial f}{\partial x} = 2x - 2y$ and $\frac{\partial f}{\partial y} = -2x + 4y$

We get $\begin{cases} 2x_0 - 2y_0 = 2 \\ -2x_0 + 4y_0 = 4 \end{cases} \Rightarrow x_0 = 4,\ y_0 = 3$

$\therefore (x_0, y_0) = (4,3)$ and $f(x_0,y_0) = f(4,3) = 10$

Hence eqn of tangent plane @ $(x_0,y_0) = (4,3)$ is

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$z = 10 + 2(x-4) + 4(y-3)$$ ✓

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page 115 #12: $z = f(x,y) = e^{2x+3y}$. $\frac{\partial f}{\partial x} = 2e^{2x+3y}$, $\frac{\partial f}{\partial y} = 3e^{2x+3y}$

$(x_0,y_0) = (0,0)$ so eqn of tangent plane is

(a) $z = 1 + 2x + 3y$

(b) $f(0.1, 0) \approx 1.2$; $f(0, 0.1) \approx 1.3$

(c) Calculator: $f(0.1, 0) = e^{0.2} = 1.221402758...$

$f(0, 0.1) = e^{0.3} = 1.349858808...$

page 115 #16(a): Estimate $(0.99e^{0.02})^8$.

let $f(x,y) = (xe^y)^8 = x^8 e^{8y}$, ~~with~~ $(x_0, y_0) = (1, 0)$

(Note; there are ∞ many choices of f)

Equation of tangent plane is $z = 1 + 8(x-1) + 8y$

$\therefore f(0.99, 0.02) \approx 1 + 8(-0.01) + 8(0.02) = 1.08$

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4 $w = \dfrac{r^2+s^2}{r^2-s^2} \Rightarrow \dfrac{\partial w}{\partial r} = \dfrac{(r^2-s^2)(2r) - (r^2+s^2)(2r)}{(r^2-s^2)^2} = \dfrac{-4rs^2}{(r^2-s^2)^2}$

$\dfrac{\partial w}{\partial s} = \dfrac{(r^2-s^2)(2s) - (r^2+s^2)(-2s)}{(r^2-s^2)^2} = \dfrac{4r^2s}{(r^2-s^2)^2}$

$\therefore r\dfrac{\partial w}{\partial r} + s\dfrac{\partial w}{\partial s} = \dfrac{-4r^2s^2}{(r^2-s^2)} + \dfrac{4r^2s^2}{(r^2-s^2)^2} = 0$

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5 $A\vec{x} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 \\ b_1x_1 + b_2x_2 \\ c_1x_1 + c_2x_2 \end{bmatrix}$

$\therefore f(x_1, x_2) = (\underbrace{a_1x_1 + a_2x_2}_{f_1}, \underbrace{b_1x_1 + b_2x_2}_{f_2}, \underbrace{c_1x_1 + c_2x_2}_{f_3})$

$\therefore Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = A$