SOLUTIONS

MA 362 - Fall 2016

Homework Set # 4

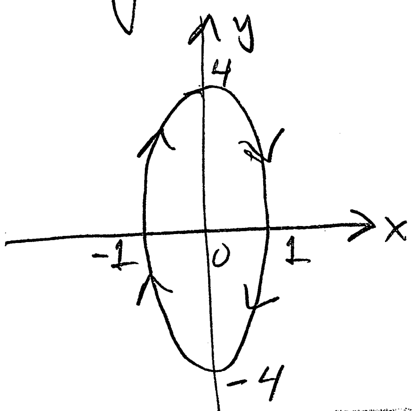
1. **(§2.4) Page 123:** # 1, 13, 17, 19, 20, 23.
2. **(§2.5) Page 132:** # 7, 10(a), 35.
3. If $g(x, y, z) = (x,\ x + y,\ x^2 + z,\ z)$ and $f(x, y, z, w) = (x^2 + z,\ y^2 - w)$, then $D(f \circ g)(1, 1, 1) = ?$
4. If $w = \sqrt{x^2 + y^2}$, with $\begin{cases} x = 3 + st \\ y = s^2 - 2t \end{cases}$, compute $\dfrac{\partial w}{\partial s}$ when $x = 3$ and $y = -4$.
5. If $w = f(x^2 - y^2, e^x)$, where $\dfrac{\partial f(u, v)}{\partial u} = \dfrac{1}{u}$ and $\dfrac{\partial f(u, v)}{\partial v} = \tan v$, express $\dfrac{\partial w}{\partial x}$ as a function of x and y.
6. The voltage V in a simple electric circuit is slowly decreasing as the battery wears out and the resistance R is slowly increasing as the resistor heats up. Given that $\dfrac{dV}{dt} = -0.01$ V/sec and $\dfrac{dR}{dt} = 0.03\ \Omega$/sec, find the rate of change of the current I when $R = 400\ \Omega$ and $V = 32$ volts.

   Remark: **Ohm's Law:** $V = IR$

Solutions

1 page 123 #1: $C: \begin{cases} x = \sin t \\ y = 4\cos t \end{cases} \quad 0 \le t \le 2\pi$

$\Rightarrow x^2 + \left(\frac{y}{4}\right)^2 = \sin^2 t + \cos^2 t = 1$ Thus C lies on the ellipse $x^2 + \frac{y^2}{4} = 1$. In fact, the ellipse is traversed exactly one in a counterclockwise direction starting at $(0,4)$:



$\frac{dy}{dt} = -4\sin t < 0$ for $0 < t < \pi$
and > 0 for $\pi < t < 2\pi$
so curve is traversed once.

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page 123 #13: $\vec{c}(t) = (t\sin t, 4t)$

$\vec{c}\,'(t) = (t\cos t + \sin t, 4)$ = tangent vector ✓

Page 123 #17: $\vec{c}(t) = (\sin 3t, \cos 3t, 2t^{5/2})$; $t = 1$

Eqn of tangent line is $\vec{r}(t) = \vec{c}(1) + (t-1)\vec{c}\,'(t)$

$\Rightarrow \vec{r}(t) = (\sin 3, \cos 3, 2) + (t-1)(3\cos 3, -3\sin 3, 5)$ ✓

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page 123 #19: $\vec{c}(t) = (t^2, t^3 - 4t, 0)$; $t_0 = 2$, $t_1 = 3$

$\vec{r}(t) = \vec{c}(2) + (t-2)\vec{c}\,'(2) = (4,0,0) + (t-2)(4,8,0)$

$\therefore \vec{r}(3) = (4,0,0) + (1)(4,8,0) = (8,8,0)$ ✓

page 123 #20: $\vec{c}(t) = (e^t, e^{-t}, \cos t)$; $t_0 = 1$, $t_1 = 2$

$\vec{r}(t) = \vec{c}(1) + (t-1)\vec{c}\,'(t) = (e, e^{-1}, \cos 1) + (t-1)(e, -e^{-1}, -\sin 1)$

$\therefore \vec{r}(2) = (e, e^{-1}, \cos 1) + (e, -e^{-1}, -\sin 1) = (2e, 0, \cos 1 - \sin 1)$ ✓

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page 123 #23: $\vec{c}(t) = (\cos t, \sin t, t^2)$

(a) speed $= \|\vec{c}\,'(4\pi)\| = \|(0, 1, 8\pi)\| = \sqrt{1 + 64\pi^2}$ ✓

(b) $\vec{c}\,'(t) = (-\sin t, \cos t, 2t)$

$\therefore \vec{c}(t) \cdot \vec{c}\,'(t) = -\cos t \sin t + \sin t \cos t + 2t^3 = 2t^3 = 0$

$\Leftrightarrow t = 0$ ✓

(c) $\vec{r}(t) = \vec{c}(4\pi) + (t - 4\pi)\vec{c}\,'(4\pi)$

$\Rightarrow \vec{r}(t) = (1, 0, 16\pi^2) + (t - 4\pi)(0, 1, 8\pi)$

(d) From part (c)
$$\begin{cases} x = 1 \\ y = t - 4\pi \\ z = 16\pi^2 + 8\pi(t - 4\pi) \end{cases}$$

$\therefore$ line intersects xy plane when $z = 0$

i.e. $16\pi^2 + 8\pi(t - 4\pi) = 0$ or $t = 2\pi$

Hence pt on line is $(1, -2\pi, 0)$ ✓

(cont'd)

Note: Another eqn of the tangent line is

$\vec{r}(t) = \vec{c}(4\pi) + t\,\vec{c}\,'(4\pi) = (1, 0, 16\pi^2) + t\,(0, 1, 8\pi)$

and here $\begin{cases} x = 1 \\ y = t \\ z = 16\pi^2 + 8\pi t \end{cases}$

Hence the line intersects xy plane when $z = 0$

so $16\pi^2 + 8\pi t \Rightarrow t = -2\pi$

$\therefore$ pt is $(1, -2\pi, 0)$ same as before

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2 page 132 #7: $f(u,v) = (\tan(u-1) - e^v,\ u^2 - v^2)$

$g(x,y) = (e^{x-y},\ x-y)$

$\therefore (f \circ g)(x,y) = f(g(x,y)) = f(e^{x-y},\ x-y)$

$\Rightarrow (f \circ g)(x,y) = \left(\tan(e^{x-y} - 1) - e^{x-y},\ e^{2(x-y)} - (x-y)^2 \right)$ ⊛

To find $D(f \circ g)$ we can have 2 solutions:

(cont'd)

Soln 1 - Use Chain Rule: $D(f\circ g)(x,y) = \left[Df(g(x,y))\right]\left[Dg(x,y)\right]$

Since $f(u,v) = (\underbrace{\tan(u-1) - e^v}_{f_1}, \underbrace{u^2 - v^2}_{f_2})$, $g(x,y) = (\underbrace{e^{x-y}}_{g_1}, \underbrace{x-y}_{g_2})$

$$\Rightarrow Df(u,v) = \begin{bmatrix} \sec^2(u-1) & -e^v \\ 2u & -2v \end{bmatrix}, \quad Dg(x,y) = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix}$$

If $(x,y) = (1,1)$ then $g(1,1) = (1,0) = (u,v)$

$\therefore D(f\circ g)(1,1) = Df(g(1,1))\, Dg(1,1) = (Df(1,0))(Dg(1,1))$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \checkmark$$

Soln 2 - Differentiate $f\circ g$ directly using ⊛ previous page

Since $(f\circ g)(x,y) = \left(\underbrace{\tan(e^{x-y} - 1) - e^{(x-y)}}_{F_1}, \underbrace{e^{2(x-y)} - (x-y)^2}_{F_2}\right)$

$$\therefore D(f\circ g)(1,1) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}_{(x,y)=(1,1)} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \checkmark$$

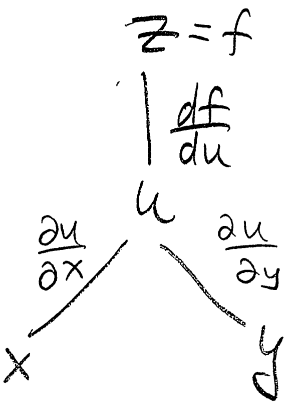
page 132 # 10(a): $T(x,y,z) = x^2+y^2+z^2$

$$\vec{\sigma}(t) = (\underset{x}{\underset{\parallel}{\cos t}}, \underset{y}{\underset{\parallel}{\sin t}}, \underset{z}{\underset{\parallel}{t}})$$

$$\frac{dT}{dt} = D(T \circ \vec{\sigma}) = \nabla T \cdot \vec{\sigma}' = (2x, 2y, 2z) \cdot (-\sin t, \cos t, 1)$$

$$= (2\cos t, 2\sin t, 2t) \cdot (-\sin t, \cos t, 1) = 2t \checkmark$$

page 132 # 35: $z = f(x-y)$ i.e., $z = f(u)$ where $u = x-y$



$$\Rightarrow \frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \left(\frac{df}{du}\right)(1)$$

$$\frac{\partial z}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = \left(\frac{df}{du}\right)(-1)$$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \left(\frac{df}{du}\right) + \left(\frac{df}{du}\right)(-1) = 0$$

$\boxed{3}$ $g(x,y,z) = (\underset{g_1}{x}, \underset{g_2}{x+y}, \underset{g_3}{x^2+z}, \underset{g_4}{z})$; $f(x,y,z,w) = (\underset{f_1}{x^2+z}, \underset{f_2}{y^2-w})$

$$D(f\circ g)(1,1,1) = \left(Df(g(1,1,1))\right)\left(Dg(1,1,1)\right) = \left(Df(1,2,2,1)\right)\left(Dg(1,1,1)\right)$$

Now $Df(x,y,z,w) = \begin{bmatrix} 2x & 0 & 1 & 0 \\ 0 & 2y & 0 & -1 \end{bmatrix}$ and

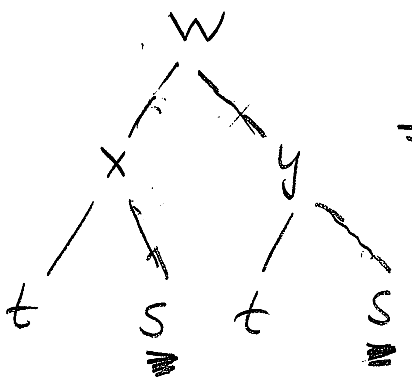
$$Dg(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2x & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore D(f\circ g)(1,1,1) = \left(Df(1,2,2,1)\right)\left(Dg(1,1,1)\right)$$

$$= \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix} \checkmark$$

$\boxed{4}$ $w=\sqrt{x^2+y^2}$, $\begin{cases} x = 3+st \\ y = s^2-2t \end{cases}$



$$\Rightarrow \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad (*)$$

Now when $x=3$ and $y=-4$

$$\Rightarrow \begin{cases} 3 = 3+st \\ -4 = s^2-2t \end{cases} \Rightarrow s=0,\ t=2$$

$$\therefore \frac{\partial w}{\partial s} = \left(\frac{x}{\sqrt{x^2+y^2}}\right)(t) + \left(\frac{y}{\sqrt{x^2+y^2}}\right)(2s)$$

@ $(s,t)=(0,2)$ and $(x,y)=(3,-4)$

$$\Rightarrow \frac{\partial w}{\partial s} = \left(\frac{3}{\sqrt{25}}\right)(2) = \frac{6}{5} \checkmark$$

$\boxed{5}$ $W = f(\underbrace{x^2-y^2}_{u}, \underbrace{e^x}_{v})$; $\frac{\partial f}{\partial u} = \frac{1}{u}$, $\frac{\partial f}{\partial v} = \tan v$

let $\begin{cases} u = x^2-y^2 \\ v = e^x \end{cases}$ $\therefore \frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$

$$= \left(\frac{1}{u}\right)(2x) + (\tan v)(e^x)$$

$$\therefore \frac{\partial w}{\partial x} = \frac{2x}{x^2-y^2} + e^x \tan(e^x) \checkmark$$

$\boxed{6}$ $I = \frac{V}{R} \Rightarrow \frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt}$

$$\frac{dI}{dt} = \left(\frac{1}{R}\right)\frac{dV}{dt} + \left(-\frac{V}{R^2}\right)\frac{dR}{dt} ;$$

$$\therefore \left.\frac{dI}{dt}\right|_{\substack{R=400 \\ V=32}} = \left(\frac{1}{400}\right)(-0.01) + \left(-\frac{32}{400^2}\right)(0.03)$$

$$= -0.000031 \text{ A/sec} \checkmark$$