SOLUTIONS

MA 362 - Fall 2016

Homework Set # 5

1. (§2.5) **Page 132:** # 5, 13(a).

2. (§2.6) **Page 142:** # 1, 2(b), 6, 8(a)(b), 9(a), 22(a)(b), 31.

3. Given the following equation relating x, y, and z, answer the questions below:

$$y^3 + x^2 - yz^3 = 4xz + 12 \quad (\#)$$

   (a) If $z = z(x, y)$ is defined implicitly by (#), compute $\dfrac{\partial z}{\partial y}$.

   (b) If $y = y(x, z)$ is defined implicitly by (#), compute $\dfrac{\partial y}{\partial x}$.

   (c) Find an equation of the tangent plane to the surface defined by (#) at the point $(x_0, y_0, z_0) = (-2, 0, 1)$.

4. Let $f(x, y, z) = x^2y + xe^{-z}$ and $\mathbf{c}(t) = \left(t^2 + t,\, t^{-1},\, t - 1\right)$.

   (a) Find the rate of change of f along the path $\mathbf{c}$ at $t = 1$.

   (b) Find the directional derivative of f in the direction of the tangent to the path $\mathbf{c}$ at $t = 1$.

5. Compute the directional derivative of $f(x, y, z) = x^2y + xe^{-z} + 10$ at $(1, -2, 0)$ in the direction from $(1, -2, 0)$ towards the origin. Is the function f increasing or decreasing?

Solutions

1 page 132 #5: $f, g : R^3 \to R$ differentiable then

$$\nabla(fg) = \left[\frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg), \frac{\partial}{\partial z}(fg)\right]_{1\times 3}$$

$$= \left[\left\{f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right\}, \left\{f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right\}, \left\{f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right\}\right]$$

$$= \left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right] + \left[g\frac{\partial f}{\partial x}, g\frac{\partial f}{\partial y}, g\frac{\partial f}{\partial z}\right]$$

$$= f\left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right] + g\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = f\nabla g + g\nabla f \checkmark$$

page 132 #13 (a): $\vec{c}(t) = (\cos t, \sin t)$; $T(x,y) = x^2 e^y - xy^3$

(a) $$\frac{dT}{dt} = \frac{d}{dt}(T \circ \vec{c}) = \frac{dT(\vec{c}(t))}{dt} = \nabla T(\vec{c}(t)) \cdot \vec{c}\,'(t)$$

$$= (2xe^y - y^3, x^2e^y - 3xy^2) \cdot (-\sin t, \cos t)$$

$$= (2xe^y - y^3)(-\sin t) + (x^2e^y - 3xy^2)(\cos t)$$, and since $x = \cos t$, $y = \sin t$

$$= \left\{2(\cos t)e^{\sin t} - \sin^3 t\right\}(-\sin t) + (\cos^2 t\, e^{\sin t} - 3\cos t \sin^2 t)(\cos t) \checkmark$$

$\boxed{2}$ page 142 #1: $f(x,y,z) = z^2x + y^3$; $(x_0,y_0,z_0) = (1,1,2)$

$\vec{u} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) \leftarrow$ already unit vector

$$\therefore D_{\vec{u}} f(1,1,2) = \nabla f(1,1,2) \cdot \vec{u} = (4,3,4) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) = \frac{10}{\sqrt{5}} = 2\sqrt{5} \checkmark$$

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page 142 #2(b): $f(x,y) = \ln\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\ln(x^2+y^2)$; $(x_0,y_0) = (1,0)$

$\vec{v} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \leftarrow$ already unit vector

$$\therefore D_{\vec{v}} f(1,0) = \nabla f(1,0) \cdot \vec{v} = \left.\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)\right|_{(1,0)} \cdot \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$$= (1,0) \cdot \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} \checkmark$$

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page 142 #6: $\underbrace{x^3 + xy + y^3}_{f(x,y)} = 11$; $(x_0,y_0) = (1,2)$

Since $\nabla f \perp$ level curves $\Rightarrow \vec{n} = \nabla f(1,2) = (5,13)$

is a normal vector to curve

(or any non-zero multiple of $\vec{n}$)

page 142 #8:

(a) $\underbrace{x^2 + 2y^2 + 3xz}_{f(x,y,z)} = 10$; $(x_0, y_0, z_0) = (1, 2, \frac{1}{3})$

$\therefore \vec{n} = \nabla f(1, 2, \frac{1}{3}) = (3, 8, 3)$

Tangent plane: $(x-1, y-2, z-\frac{1}{3}) \cdot (3,8,3) = 0$

or $3(x-1) + 8(y-2) + 3(z - \frac{1}{3}) = 0$ ✓

(b) $\underbrace{y^2 - x^2}_{f(x,y,z)} = 3$; $(x_0, y_0, z_0) = (1, 2, 8)$

$\therefore \vec{n} = \nabla f(1,2,8) = (-2, 4, 0)$

Tangent plane: $(x-1, y-2, z-8) \cdot (-2, 4, 0) = 0$

or $-2(x-1) + 4(y-2) = 0$ ✓

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page 142 #9(a): $z = \underbrace{x^3 + y^3 - 6xy}_{f(x,y)}$; $(x_0, y_0, z_0) = (1, 2, -3)$

Old way: $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$z = -3 - 9(x-1) + 6(y-2)$ ✓

same

New way: let $F(x,y,z) = x^3 + y^3 - 6xy - z$; then surface is $F(x,y,z) = 0$

$\therefore \vec{n} = \nabla F(1,2,-3) = (-9, 6, -1)$

$\therefore$ Tangent plane is $(x-1, y-2, z+3) \cdot (-9, 6, -1) = 0$

page 142 #22 : $T(x,y,z) = e^{-x^2 - 2y^2 - 3z^2}$

(a) T decreases fastest in direction $\vec{v} = -\nabla T(1,1,1)$

i.e. $\vec{v} = -\left(-2e^{-6}, -4e^{-6}, -6e^{-6}\right) = 2e^{6}(1,2,3)$ ✓

(or any positive multiple of $\vec{v}$)

(b) Line from $(1,1,1)$ in direction $\vec{v}$ is

$\vec{r}(t) = (1,1,1) + t\vec{v}$, but speed $= \|\vec{r}\,'(t)\| = \|\vec{v}\| = 2e^{6}\sqrt{14}$

Now we want speed to be e^8, so we need to adjust $\vec{v}$:

Hence $\vec{r}(t) = (1,1,1) + t\left\{e^{8}\dfrac{\vec{v}}{\|\vec{v}\|}\right\}$ (*)

Note: $\|\vec{r}\,'(t)\| = \left\| e^{8}\dfrac{\vec{v}}{\|\vec{v}\|}\right\| = e^{8}\left\|\dfrac{\vec{v}}{\|\vec{v}\|}\right\| = e^{8}$

Now $\left.\dfrac{d}{dt} T(\vec{r}(t))\right|_{t=0} = \nabla T(\vec{r}(0)) \cdot \vec{r}\,'(0)$

$= \nabla T(1,1,1) \cdot \left\{e^{8}\dfrac{\vec{v}}{\|\vec{v}\|}\right\} = \left(-2e^{-6}, -4e^{-6}, -6e^{-6}\right) \cdot \left\{e^{8}\dfrac{\vec{v}}{\|\vec{v}\|}\right\}$

$= \left(-2e^{2}, -4e^{2}, -6e^{2}\right) \cdot \dfrac{\vec{v}}{\|\vec{v}\|} = \left(-2e^{2}, -4e^{2}, -6e^{2}\right) \cdot \left\{\dfrac{2e^{6}(1,2,3)}{2e^{6}\sqrt{14}}\right\}$

$= \dfrac{-28e^{2}}{\sqrt{14}}$ $\therefore$ Temp decreasing at $\dfrac{28e^{2}}{\sqrt{14}}$ deg/sec page 142 #31: Ejected from surface $\overbrace{x^2+y^2-z^2}^{f(x,y,z)}=-1$ @ $(1,1,\sqrt{3})$ normal directed towards xy plane to surface at $t=0$ with speed of 10 units/sec.

the normal line is $\vec{r}(t)=(1,1,\sqrt{3})+t\,\nabla f(1,1,\sqrt{3})$

$\therefore \vec{r}(t)=(1,1,\sqrt{3})+t(2,2,-2\sqrt{3})$

Since $\|\vec{r}\,'(0)\|=\|(2,2,-2\sqrt{3})\|=\sqrt{4+4+12}=\sqrt{20}$

but we want $\|\vec{r}\,'(0)\|=10$, so rescale time parameter t:

let $\vec{r}(t)=(1,1,\sqrt{3})+\left(\frac{10}{\sqrt{20}}\right)t(2,2,-2\sqrt{3})$ ✓

Thus it crosses xy-plane when z-component is 0

Hence $$\sqrt{3}+\left(\frac{10}{\sqrt{20}}\right)t(-2\sqrt{3})=0$$

$\Rightarrow \; t=\frac{1}{\sqrt{20}}$ ✓ and the point is

$$\vec{r}\left(\frac{1}{\sqrt{20}}\right)=(1,1,\sqrt{3})+\left(\frac{10}{\sqrt{20}}\right)\left(\frac{1}{\sqrt{20}}\right)(2,2,-2\sqrt{3})$$

$$=(2,2,0)$$ ✓

$\boxed{3}$ $\underbrace{y^3+x^2y z^3-4xz}_{F(x,y,z)}=12$

ⓐ Implicit diff $\Rightarrow$ $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z} = -\dfrac{(3y^2 - z^3)}{(-3yz^2-4x)}$ ✓

ⓑ Implicit diff $\Rightarrow$ $\dfrac{\partial y}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial y} = -\dfrac{(2x-4z)}{(3y^2-z^3)}$ ✓

ⓒ Tangent plane: $\vec{n} = \nabla F(-2,0,1) = (-8,-1,8)$

$\therefore$ $(x+2, y-0, z-1)\cdot(-8,-1,8)=0$

or $-8(x+2)-y+8(z-1)$ ✓

$\boxed{4}$ $f(x,y,z)=x^2y+xe^{-z}$, $\vec{c}(t)=(t^2+t, \frac{1}{t}, t-1)$

ⓐ $\left.\dfrac{d}{dt}f(\vec{c}(t))\right|_{t=1} = \nabla f(\vec{c}(1))\cdot\vec{c}\,'(1) = \nabla f(2,1,0)\cdot(3,-1,1)$

$= (5,4,-2)\cdot(3,-1,1) = 9$ ✓

ⓑ Direction is $\vec{c}\,'(1)=(3,-1,1)$ $\therefore$ $\vec{u} = \dfrac{\vec{c}\,'(1)}{\|\vec{c}\,'(1)\|} = \dfrac{(3,-1,1)}{\sqrt{11}}$

So $D_{\vec{u}}f(2,1,0) = \nabla f(2,1,0)\cdot\vec{u} = \dfrac{(5,4,-2)\cdot(3,-1,1)}{\sqrt{11}} = \dfrac{9}{\sqrt{11}}$ ✓

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$\boxed{5}$ $f(x,y,z) = x^2y + xe^{-z}$; $P(1,-2,0)$

direction is $\overrightarrow{PQ} = (0,0,0) - (1,-2,0) = (-1,2,0)$ Q(0,0,0)

$\therefore \vec{u} = \dfrac{(-1,2,0)}{\sqrt{5}}$ ✓

So $D_{\vec{u}} f(1,-2,0) = \nabla f(1,-2,0) \cdot \vec{u} = (-3,1,1) \cdot \dfrac{(-1,2,0)}{\sqrt{5}}$

$= \dfrac{5}{\sqrt{5}} = \sqrt{5} > 0 \Rightarrow f$ is increasing at $(1,-2,0)$ in direction $\overrightarrow{PQ}$.