SOLUTIONS

MA 362 - Fall 2016

Homework Set # 6

1. Show that the ellipsoid $6x^2 + y^2 + 6y + z^2 = 15$ and the surface $z = x^2 e^y - 4$ are tangent at their point of intersection $(1, 0, -3)$.

2. (§3.1) **Page 156:** # 4, 11(a), 22, 25.

3. (§3.2) **Page 165:** # 1.

4. Show that if x and y are close to 0, then $\dfrac{\cos x}{\cos y} \approx 1 - \dfrac{x^2}{2} + \dfrac{y^2}{2}$.

5. (§3.3) **Page 182:** # 4, 17, 20, 28, 39.

6. Find and classify all critical points of $f(x, y) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - 4xy + y^2 + 10$.

7. (§3.4) **Page 201:** # 4, 21.

Solutions

1. Only need to show the tangent planes to both surfaces @ $(1,0,-3)$ are the same:

Surface 1: $\underbrace{6x^2+y^2+6y+z^2}_{F(x,y,z)}=15$

$\therefore\ \vec{n}=\nabla F(1,0,-3)=(12,6,-6)$

So tangent plane is $(x-1,\ y-0,\ z+3)\cdot\vec{n}=0$

or $12(x-1)+6y-6(z+3)=0$ ⊛

Surface 2: $\underbrace{z-x^2e^y}_{F(x,y,z)}=-4$

$\therefore\ \vec{n}=\nabla F(1,0,-3)=(-2,-1,1)$

So tangent plane is $(x-1,\ y-0,\ z+3)\cdot\vec{n}=0$

or $-2(x-1)-y+(z+3)=0$

This is exactly the same plane as ⊛

$\boxed{2}$ **page 156 #4:** $f(x,y) = e^{-xy^2} + y^3x^4$

$f_x = -y^2 e^{-xy^2} + 4y^3x^3$; $f_y = -2xye^{-xy^2} + 3y^2x^4$

$\therefore$ $f_{xx} = y^3 e^{-xy} + 12y^3x^2$; $f_{xy} = \{2xy^3e^{-xy^2} - 2ye^{-xy^2}\} + 12y^2x^3$

$f_{yx} = \{2xy^3e^{-xy^2} - 2ye^{-xy^2}\} + 12y^2x^3$

$f_{yy} = \{4x^2y^2e^{-xy^2} - 2xe^{-xy^2}\} + 6yx^4$

Thus $f_{xy} = f_{yx}$

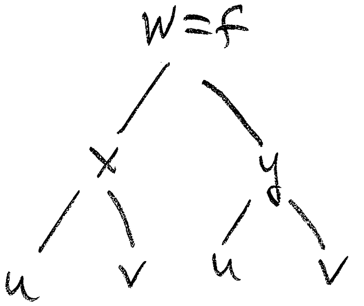
Page 156 # 11 (a): $f(x,t) = \sin(x - ct)$

$f_x = \cos(x-ct)$; $f_{xx} = -\sin(x-ct)$

$f_t = \{\cos(x-ct)\}\{-c\}$; $f_{tt} = \{-\sin(x-ct)\}\{-c\}\{-c\}$

$\therefore$ $f_{xx} = \frac{1}{c^2} f_{tt}$

page 156 # 22: $w = f(x, y)$ where $\begin{cases} x = u + v \\ y = u - v \end{cases}$



$$\Rightarrow \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial w}{\partial x}(1) + \frac{\partial w}{\partial y}(-1)$$

$$\therefore \quad \frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)$$

use Chain Rule again

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)\left\{\frac{\partial x}{\partial u}\right\} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)\left\{\frac{\partial y}{\partial u}\right\}$$

$$= \left[\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial x \partial y}\right](1) + \left[\frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial y^2}\right](1)$$

$$= \frac{\partial^2 w}{\partial x^2} \underbrace{- \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial x}}_{\text{equality of mixed partials}} - \frac{\partial^2 w}{\partial y^2}$$

$$\Rightarrow \quad \frac{\partial^2 w}{\partial u \, \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

page 156 # 25: $u(x,y) = x^3 - 3xy^2$

since $u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$

$u_y = -6xy$, $u_{yy} = -6x$

$\Rightarrow \underbrace{u_{xx} + u_{yy} = 0}_{\text{Laplace's Equation}}$. Hence $u(x,y) = x^3 - 3xy^2$ is a harmonic function

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page 165 # 1: $f(x,y) = e^{x+y}$; $(x_0,y_0) = (0,0)$

$\nabla f(0,0) = (1,1)$

1st order Taylor Formula: $f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot \vec{h} + [R_1]$

since $\vec{x}_0 = (0,0)$, $\vec{h} = (h_1, h_2)$ $\Rightarrow f(h_1,h_2) = 1 + (1,1) \cdot (h_1,h_2) + [R_1]$

so $f(h_1,h_2) = 1 + h_1 + h_2 + [R_1]$

Or, it could be written in equivalent form:

$f(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) + [R_1]$

here let $\vec{x} = (x,y)$; $\vec{x}_0 = (0,0)$

$\therefore f(x,y) = 1 + (1,1) \cdot (x-0, y-0) + [R_1]$

so $f(x,y) = 1 + x + y + [R_1]$

(cont'd)

2^{nd} Order Taylor Formula:

$$f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot\vec{h} + \frac{1}{2}\vec{h}\, Hf(\vec{x}_0)\,\vec{h}^T + [R_2] \quad (*)$$

Now $\vec{x}_0 = (0,0)$, $\nabla f(\vec{x}_0) = \nabla f(0,0) = 1$

$\vec{h} = (h_1, h_2)$

and Hessian $Hf(\vec{x}_0) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{\vec{x}_0} = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{bmatrix}_{\vec{x}_0=(0,0)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore$ $(*)$ becomes

$$f(h_1,h_2) = 1 + h_1 + h_2 + \frac{1}{2}(h_1, h_2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + [R_2]$$

or

$$f(h_1,h_2) = 1 + h_1 + h_2 + \frac{1}{2}\left(h_1^2 + 2h_1h_2 + h_2\right)^2 + [R_2]$$

[4] Show $\frac{\cos x}{\cos y} \approx 1 - \frac{x^2}{2} + \frac{y^2}{2}$ for x, y close to 0. ⑥

~~Note~~ that since there are quadratic terms we need to use 2^{nd} order Taylor Formula approximation:

$$f(\vec{x}) \approx f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) + \frac{1}{2}(\vec{x}-\vec{x}_0)\, Hf(\vec{x}_0)\, (\vec{x}-\vec{x}_0)^T$$

$\vec{x} = (x, y)$; $\vec{x}_0 = (x_0, y_0) = (0,0)$

$$\nabla f(0,0) = \left(-\frac{\sin x}{\cos y}, \cos x \sec y \tan y\right)_{(0,0)} = (0,0)$$

$$Hf(0,0) = \begin{bmatrix} -\frac{\cos x}{\cos y} & \{-\sin x \sec y \tan y\} \\ \{-\sin x \sec y \tan y\} & \left\{\frac{(\cos x)[\cos^3 y + 2\cos y \sin^2 y]}{\cos^4 y}\right\} \end{bmatrix}_{(0,0)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore f(x,y) \approx 1 + (0,0) \cdot (x-0, y-0) + \frac{1}{2}(x-0, y-0)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

Hence

$$\frac{\cos x}{\cos y} \approx 1 + \frac{1}{2}(-x^2 + y^2)$$ ✓

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page 182 #4: $f(x,y) = x^2 + y^2 + 3xy$

$\nabla f(x,y) = (2x+3y, 2y+3x) = (0,0)$

$\Rightarrow \begin{cases} 2x+3y=0 \\ 2y+3x=0 \end{cases} \Rightarrow$ (0,0) only critical pt.

Hessian $Hf = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

$\therefore$ $\det Hf = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5 < 0 \Rightarrow (0,0)$ is a saddle pt of f

page 182 #17: $f(x,y) = 8y^3 + 12x^2 - 24xy$

$\nabla f = (24x - 24y, 24y^2 - 24x) = (0,0)$

$\Rightarrow \begin{cases} 24x - 24y = 0 & ① \\ 24y^2 - 24x = 0 & ② \end{cases}$

① $\Rightarrow$ $y=x$ plug into ② to get $24(x^2) - 24x = 0$

or $x^2 - x = 0$

so $x=0$ or $x=1$

$\therefore$ ~~Critical~~ pts are $(0,0)$, $(1,1)$

(cont'd)

Now $Hf(x,y) = \begin{bmatrix} 24 & -24 \\ -24 & 48y \end{bmatrix} = (24)\begin{bmatrix} 1 & -1 \\ -1 & 2y \end{bmatrix}$

$\boxed{(0,0)} \Rightarrow \det Hf(0,0) = (24)\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -24 < 0$

$\Rightarrow$ $(0,0)$ <u>saddle pt</u> of f

$\boxed{(1,1)} \Rightarrow \det Hf(1,1) = (24)\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 24 > 0.$

$\therefore$ $(1,1)$ is either local max/min

Need to next check $f_{xx}(1,1) = 24 > 0$

$\Rightarrow$ $(1,1)$ is <u>local min</u> of f

Page 182 # 20: Given $\nabla f(4,2) = (0,0)$.

Know $Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

(a) $\det H(4,2) = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = -4 < 0 \Rightarrow (4,2)$ is saddle pt

(b) $\det H(4,2) = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} = 7 > 0 \Rightarrow (4,2)$ is either local max or min

Need to check $f_{xx}(4,2) = 2 > 0 \Rightarrow (4,2)$ is local min

(c) $\det H(4,2) = \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} = -7 < 0 \Rightarrow (4,2)$ is saddle pt page 182 #28: Find point on plane $2x-y+2z=20$ nearest $(0,0,0)$

Let (x,y,z) be any pt on the plane, then we need minimize $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$. It is equivalent to minimize its square. Hence we want

$$\boxed{\begin{array}{l} \min\ f(x,y,z)=x^2+y^2+z^2 \\ \text{s.t. } 2x-y+2z=20 \end{array}} \quad (*)$$

Soln 1 - Since $y=2x+2z-20$

$$\Rightarrow \quad f(x,z)=x^2+(2x+2z-20)^2+z^2$$

$$\nabla f(x,z)=\big(2x+2(2x+2z-20)(2),\ 2(2x+2z-20)(2)+2z\big)=(0,0)$$

$$\Rightarrow \begin{cases} 2x+4(2x+2z-20)=0 \\ 4(2x+2z-20)+2z=0 \end{cases} \Rightarrow \begin{cases} 5x+4z=40 \\ 4x+5z=40 \end{cases}$$

$\Rightarrow z=\frac{40}{9}$, $x=\frac{40}{9}$ and since $y=2x+2z-20=-\frac{20}{9}$

$\therefore \left(\frac{40}{9}, -\frac{20}{9}, \frac{40}{9}\right)$ is the only possible answer and we know a soln exists. Hence the answer is $\boxed{(40/9,\ -20/9,\ 40/9)}$. min distance $=\frac{60}{9}$

(cont'd)

Soln 2: Use Lagrange Multiplier Rule

From ⊛ we check pts where $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 20 \end{cases}$

where $g(x,y,z) = 2x - y + 2z$

$$\therefore \begin{cases} (2x, 2y, 2z) = \lambda(2, -1, 2) \\ 2x - y + 2z = 20 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 2\lambda & (1) \\ 2y = -\lambda & (2) \\ 2z = 2\lambda & (3) \\ 2x - y + 2z = 20 & (4) \end{cases}$$

$\therefore$ (1) and (3) $\Rightarrow$ $x = z$

(2) and (3) $\Rightarrow$ $y = -\frac{1}{2} z$

$\therefore$ (3) $\Rightarrow$ $2(z) - \left(-\frac{1}{2} z\right) + 2z = 20 \Rightarrow z = \frac{40}{9}$

and $x = z = \frac{40}{9}$ and $x = -\frac{1}{2} z = -\frac{20}{9}$

$\therefore$ point is $\left(\frac{40}{9}, -\frac{20}{9}, \frac{40}{9}\right)$, same as in Soln 1

page 182 #39: Extremize $f(x,y) = x^2 + xy + y^2$
s.t. $x^2 + y^2 \leq 1$

Find admissible ~~critical~~ pts: $\nabla f = \vec{0}$

$\Rightarrow (2x+y, y+2y) = (0,0) \Rightarrow$ $x=0, y=0$ is only ~~critical~~ pt. inside D

where $D: x^2 + y^2 \leq 1$. Now consider $\partial D: x^2 + y^2 = 1$

let $x = \cos\theta$
$y = \sin\theta$ then on ∂D, the function f is

$$f = \cos^2\theta + \cos\theta \sin\theta + \sin^2\theta = 1 + \cos\theta \sin\theta$$

$$\therefore \frac{df}{d\theta} = \cos^2\theta - \sin^2\theta = 0 \Rightarrow \tan\theta = \pm 1$$

$\therefore \theta = \pm \frac{\pi}{4}$, hence $(x,y) = \left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$

Table of Values:

(x,y)	$f(x,y) = x^2 + y^2 + y^2$
$(0,0)$	0 ← Abs min value
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\frac{3}{2}$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\frac{3}{2}$

abs max value

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$$f(x,y) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - 4xy + y^2 + 10$$

$$\nabla f(x,y) = \left(x^3 - 2x^2 - 4y, -4x + 2y\right) = (0,0)$$

$$\begin{cases} x^3 - 2x^2 - 4y = 0 & \text{(1)} \\ -4x + 2y = 0 & \text{(2)} \end{cases}$$

(2) $\Rightarrow$ $y = 2x$ plug into (1) to get $x^3 - 2x^2 - 4(2x) = 0$

$\Rightarrow x^3 - 2x^2 - 8x = 0$ or $x(x^2 - 2x - 8) = 0$

$$x(x-4)(x+2) = 0$$

$\therefore$ $x = 0$, $x = 4$, $x = -2$

So 3 critical pts are $(0,0)$, $(4,8)$, $(-2,-4)$ (since $y = 2x$).

Now need to use 2nd Derivative Test on these 3 critical pts. We'll need the Hessian of f:

$$Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} (3x^2 - 4x) & -4 \\ -4 & 2 \end{bmatrix}$$

(cont'd)

Since $Hf(x,y) = \begin{bmatrix} (3x^2-4x) & -4 \\ -4 & 2 \end{bmatrix}$,

At $\boxed{(0,0)}$ $\det Hf(0,0) = \begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} = -16 < 0$

$\Rightarrow f(0,0)$ is a <u>saddle pt</u> of f

(i.e., $(0,0)$ is a critical pt, but <u>not</u> local max/min)

(like an inflection pt in Calculus I)

At $\boxed{(4,8)}$ $\det Hf(4,8) = \begin{vmatrix} 16 & -4 \\ -4 & 2 \end{vmatrix} = 16 > 0$ $\therefore$ $(4,8)$ is either local max/min

Need to check $f_{xx}(4,8) = 16 > 0$ $\therefore$ $(4,8)$ is a <u>local min</u>

At $\boxed{(-2,-4)}$ $\det Hf(-2,-4) = \begin{vmatrix} 24 & -4 \\ -4 & 2 \end{vmatrix} = 32 > 0$

$\therefore (-2,-4)$ is either local max/min

Need to check sign of $f_{xx}(-2,-4) = 24 > 0$

$\therefore$ $(-2,-4)$ is a <u>local min</u>

7 page 201 #4: Extremize $f(x,y) = x - y$

s.t. $\underbrace{x^2 - y^2}_{g(x,y)} = 2$

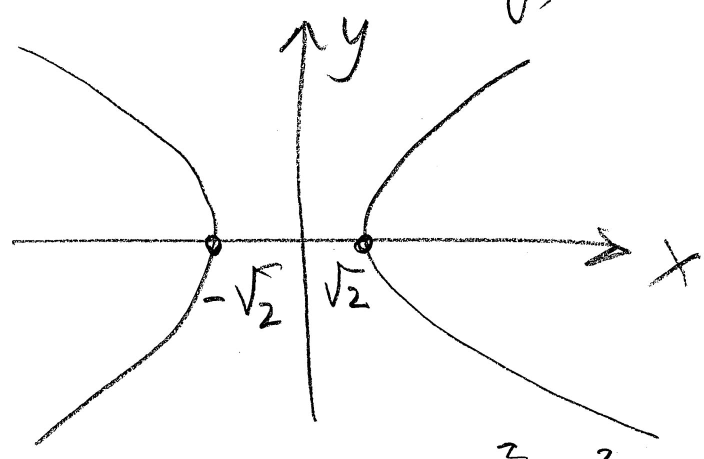
LMR $\Rightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g = 2 \end{cases} \Rightarrow \begin{cases} (1,-1) = \lambda(2x, -2y) \\ x^2 - y^2 = 2 \end{cases}$

$\Rightarrow \begin{cases} 1 = 2\lambda x & ① \\ -1 = -2\lambda y & ② \\ x^2 - y^2 = 2 & ③ \end{cases}$

① & ② $\Rightarrow x = y$ but ③ says that $x^2 - y^2 = 2$

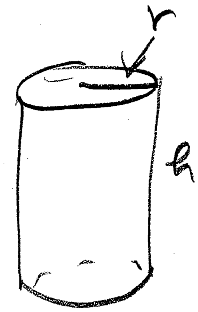
this is impossible.

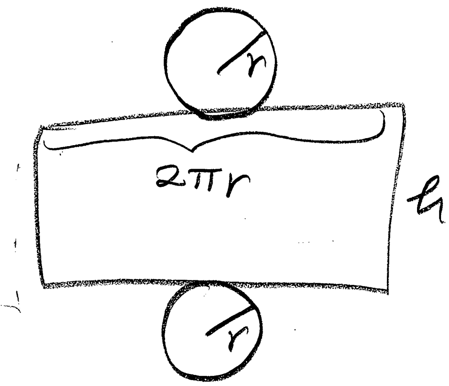
Note that the original problem has <u>no</u> solution since $f(x,y) = x - y$ can be arbitrarily large or small for (x,y) on the constraint:



$x^2 - y^2 = 2$

page 201 #21:





Minimize $S = 2(\pi r^2) + 2\pi r h$

s.t. $\underbrace{\pi r^2 h}_{g(r,h)} = 1$

$$\text{LMR} \Rightarrow \begin{cases} \nabla S = \lambda \nabla g \\ g(r,h) = 1 \end{cases} \Rightarrow \begin{cases} (4\pi r + 2\pi h,\ 2\pi r) = \lambda\,(2\pi r h,\ \pi r^2) \\ \pi r^2 h = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 4\pi r + 2\pi h = \lambda\,(2\pi r h) \\ 2\pi r = \lambda \pi r^2 \\ \pi r^2 h = 1 \end{cases} \quad \text{i.e.}$$

$$\begin{cases} 2r + h = \lambda r h \quad ① \\ 2r = \lambda r^2 \quad ② \Rightarrow \lambda = \frac{2}{r} \quad (\text{since } r \neq 0) \\ \pi r^2 h = 1 \quad ③ \end{cases}$$

$\therefore$ ① $\Rightarrow 2r + h = \left(\frac{2}{r}\right) r h = 2h \Rightarrow 2r = h$

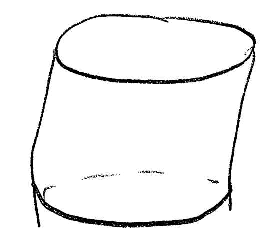
(cont'd)

Since $h = 2r$, (3) $\Rightarrow \pi r^2 (2r) = 1$

$$\Rightarrow \quad r = \left(\frac{1}{2\pi}\right)^{1/3}$$

Thus can has $r = \left(\frac{1}{2\pi}\right)^{1/3}$ and $h = 2\left(\frac{1}{2\pi}\right)^{1/3}$

i.e. can should look like:



diameter = height