SOLUTIONS

MA 362 - Fall 2016

Homework Set # 7

1. Maximize $f(x,y,z) = z - 2x + y$ subject to the two constraints $x^2 + y^2 = 8$ and $y - z = 1$.

2. Given the system of equations $\begin{cases} x + 2z + t^2 + u^2 + e^{2y} = 5 \\ x^2 + u + z + ty + y^3 = 0 \end{cases}$, is it possible to solve for the functions $u = u(x,z,t)$ and $y = y(x,z,t)$ in a neighborhood of $(x_0, y_0, z_0, t_0, u_0) = (0,0,1,1,-1)$?

3. Consider the lamina (thin plate in $\mathbb{R}^2$) bounded below by $y = x^2 - 1$ and bounded above by $y = 3$. If the temperature at (x,y) is $T(x,y) = xy - x^2 + 10$, where is the coldest temperature on the plate? What is this coldest temperature?

4. (§4.1) Page 227: # 23.

5. Find the length of these curves:
   (a) $\mathbf{r}(t) = (2t)\,\mathbf{i} + t^2\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}, \;\; 0 \le t \le 1.$
   (b) $y = \frac{1}{3}\,x^{\frac{3}{2}}$ from $(0,0)$ to $(9,9)$.

6. If $f, g, \phi : \mathbb{R} \longrightarrow \mathbb{R}$ are C^1 real-valued functions and $\mathbf{c}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, prove that

$$\frac{d}{dt}\Big(\phi(t)\,\mathbf{c}(t)\Big) = \phi(t)\,\mathbf{c}'(t) + \phi'(t)\,\mathbf{c}(t)$$

Solutions

1 max $f(x,y,z) = z - 2x + y$

s.t. $\underbrace{x^2 + y^2}_{g_1(x,y)} = 8$

$\underbrace{y - z}_{g_2(x,y)} = 1$

$\therefore \begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 8 \\ g_2 = 1 \end{cases}$

$\Rightarrow \begin{cases} (-2, 1, 1) = \lambda_1 (2x, 2y, 0) + \lambda_2 (0, 1, -1) \\ g_1 = 8 \\ g_2 = 1 \end{cases}$

$\Rightarrow \begin{cases} -2 = 2\lambda_1 x & (1) \\ 1 = 2\lambda_1 y + \lambda_2 & (2) \\ 1 = -\lambda_2 & (3) \\ x^2 + y^2 = 8 & (4) \\ y - z = 1 & (5) \end{cases}$

(2), (3) $\Rightarrow 1 = 2\lambda_1 y - 1 \Rightarrow \boxed{1 = \lambda_1 y}$

hence $\lambda_1 \neq 0$, $y \neq 0$

$\therefore \lambda_1 = \frac{1}{y}$ ✓

(1) $\Rightarrow -2 = 2\left(\frac{1}{y}\right)x \Rightarrow \boxed{-y = x}$ hence (4) $\Rightarrow y^2 + y^2 = 8$

$y = \pm 2$

(5) $\Rightarrow z = y - 1$

$\therefore$ Possible pts are

$(2, -2, -3)$ and $(-2, 2, 1)$

so

(x, y, z)	$f(x,y,z) = z - 2x + y$
$(2, -2, -3)$	-9
$(-2, 2, 1)$	7 ← max value of f

2. Let $F_1(x,y,z,t,u) = x + 2z + t^2 + u^2 + e^{2y} - 5$

$F_2(x,y,z,t,u) = x^2 + u + z + ty + y^3$

If $\Delta = \det \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial y} \end{bmatrix} \neq 0$ @ $(x_0, y_0, z_0, t_0, u_0) = (0,0,1,1,-1)$

then by the Implicit Function Thm we can solve for u, y as a function of x, z, t in the system

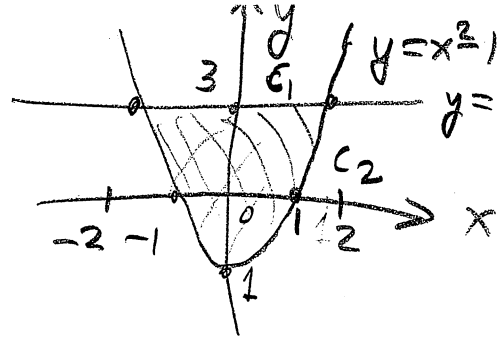
$$\begin{cases} F_1(x,y,z,t,u) = 0 \\ F_2(x,y,z,t,u) = 0 \end{cases} \quad (*)$$

Now

$$\Delta = \begin{vmatrix} (2u) & (2e^{2y}) \\ 1 & (t+3y^2) \end{vmatrix}_{(x_0,y_0,z_0,t_0,u_0) = (0,0,1,1,-1)} = \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = -4 \neq 0$$

Thus **YES** it is possible to solve for u, y in $(*)$.

[3] $T(x,y) = xy - x^2 + 10$



$\nabla T = (y-2x, x) = (0,0) \Rightarrow x=0, y=0$

$\therefore$ $\boxed{(0,0)}$ ✓ admissible critical pt

On C_1: $y=3,\ -2 \le x \le 2 \Rightarrow T = 3x - x^2 + 10 \Rightarrow \frac{dT}{dx} = 3 - 2x = 0$
$x = 3/2$

$\therefore$ get $\boxed{(\frac{2}{3}, 3)}$ ✓ and endpts $\boxed{(-2,3), (2,3)}$ ✓

On C_2: $y = x^2 - 1,\ -2 \le x \le 2 \Rightarrow T = x(x^2-1) - x^2 + 10 = x^3 - x - x^2 + 10$

$\Rightarrow \frac{dT}{dx} = 3x^2 - 1 - 2x = (3x+1)(x-1) = 0 \Rightarrow x = -1/3,\ x = 1$

$\therefore$ get $\boxed{(-\frac{1}{3}, -\frac{8}{9}), (1,0)}$ ✓ and endpts $\boxed{(-2,3), (2,3)}$ ✓

Table:

(x,y)	$T(x,y) = xy - x^2 + 10$
$(0,0)$	10
$(\frac{2}{3}, 3)$	$104/9 \approx 11.55\ldots$
$(-2,3)$	0 ⟵ Abs. min
$(2,3)$	12
$(-1/3, -8/9)$	$275/27 \approx 10.185\ldots$
$(1,0)$	9

$\therefore$ Coldest temp is $0°$ and it occurs at $(-2,3)$

[4] page 221 #23: $\vec{c}'(t) = (t, e^t, t^2)$

$$\Rightarrow \vec{c}(t) = \left(\frac{t^2}{2} + c_1,\ e^t + c_2,\ \frac{t^3}{3} + C_3\right)$$

$$(0,-5,1) = \vec{c}(0) = (c_1,\ 1+c_2,\ c_3) \Rightarrow c_1 = 0,\ c_2 = -6,\ c_3 = 1$$

$$\therefore\ \vec{c}(t) = \left(\frac{t^2}{2},\ e^t - 6,\ \frac{t^3}{3} + 1\right)\ ✓$$

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[5] (a) $\vec{r}(t) = (2t,\ t^2,\ \frac{1}{3}t^3)$, $0 \le t \le 1$

$$\Rightarrow L = \int_0^1 \|\vec{r}'(t)\| dt = \int_0^1 \sqrt{4 + 4t^2 + t^4}\, dt = \int_0^1 \sqrt{(2+t^2)^2}\, dt$$

$$= \int_0^1 |2+t^2|\, dt = \int_0^1 (2+t^2)\, dt = \boxed{\frac{7}{3}}$$

b) $y = \frac{1}{3}x^{\frac{3}{2}}$; $(1, \frac{1}{3})$ to $(9,9)$

Parametrize curve as follows $\vec{c}(t) = (t,\ \frac{1}{3}t^{3/2})$

$1 \le t \le 9$

$$\therefore\ L = \int_1^9 \|\vec{r}'(t)\| dt = \int_1^9 \sqrt{1 + \frac{t}{4}}\, dt = \boxed{\frac{1}{3}\left(13^{3/2} - 5^{3/2}\right)}$$

6 $\vec{c}(t) = (f(t), g(t))$ ⑤

$\Rightarrow \phi(t)\vec{c}(t) = (\phi(t)f(t), \phi(t)g(t))$

$\therefore \frac{d}{dt}\left(\phi(t)\vec{c}(t)\right) = \left(\{\phi f' + \phi' f\}, \{\phi g' + \phi' g\}\right)$

$= \phi(f', g') + \phi'(f, g)$

$= \phi(t)\vec{c}\,'(t) + \phi'(t)\vec{c}(t)$ ∎