SOLUTIONS

MA 362 - Fall 2016

Homework Set # 8

1. (§4.2) **Page 234:** # 13.

2. An object travels along the path $\mathbf{c}(t) = \left(t,\ \frac{2}{3}t^{\frac{3}{2}}\right)$, for $0 \le t \le 1$. Find the arc length function $s(t)$ and use it to reparameterize the path so that the object's speed is a constant 1 unit/sec for all t.

3. (§4.3) **Page 243:** # 5, 9, 16, 21.

4. Given that $\mathbf{F}(x,y) = \left(\{1 + 2x\ln y\},\ \left\{\frac{x^2}{y} + 2y\right\}\right) = \nabla\phi(x,y)$, find the potential function $\phi(x,y)$ which satisfies $\phi(-1,1) = 4$.

5. (§4.4) **Page 258:** # 2, 3, 14, 24.

6. Prove the **Basic Identity** #11 on Page 255:
   If $f : U \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ is of class C^2, prove that $\nabla \times (\nabla f) = \mathbf{0}$.

$\boxed{1}$ page 234 #13 : $\vec{c}(t) = (2t, t^2, \ln t)$, $1 \le t \le 2$

$$L = \int_1^2 \|\vec{c}'(t)\| dt = \int_1^2 \sqrt{\left(\tfrac{1}{t} + 2t\right)^2}\, dt = \int_1^2 \left|\tfrac{1}{t} + 2t\right| dt$$

$$= \int_1^2 \left(\tfrac{1}{t} + 2t\right) = \ln 2 + 3 \checkmark$$

$\boxed{2}$ $\vec{c}(t) = \left(t, \frac{2}{3} t^{\frac{3}{2}}\right)$, $0 \le t \le 1$

$$\underbrace{s = s(t) = \int_0^t \|\vec{c}'(u)\| du = \int_0^t \sqrt{1+u}\, du = \frac{2}{3}\left\{(1+t)^{3/2} - 1\right\}}_{\text{Solve for } t \text{ in terms of } s}$$

$$\Rightarrow t = \left(\tfrac{3}{2}s + 1\right)^{2/3} - 1$$

$$\therefore \vec{c}(s) = \left(\left\{\left(\tfrac{3}{2}s+1\right)^{2/3} - 1\right\}, \frac{2}{3}\left\{\left(\tfrac{3}{2}s+1\right)^{2/3} - 1\right\}^{3/2} \right) \checkmark$$

or if you prefer the (dummy) variable t, then

$$\vec{c}(t) = \left(\left\{\left(\tfrac{3}{2}t+1\right)^{2/3} - 1\right\}, \frac{2}{3}\left\{\left(\tfrac{3}{2}t+1\right)^{2/3} - 1\right\}^{3/2} \right) \checkmark$$

The object now has speed $= 1$ unit/sec

3 page 243 #5: $\vec{F}(x,y) = (2y, x)$. Instead of making a huge table, consider what this vector field looks like on various lines $y = mx$:

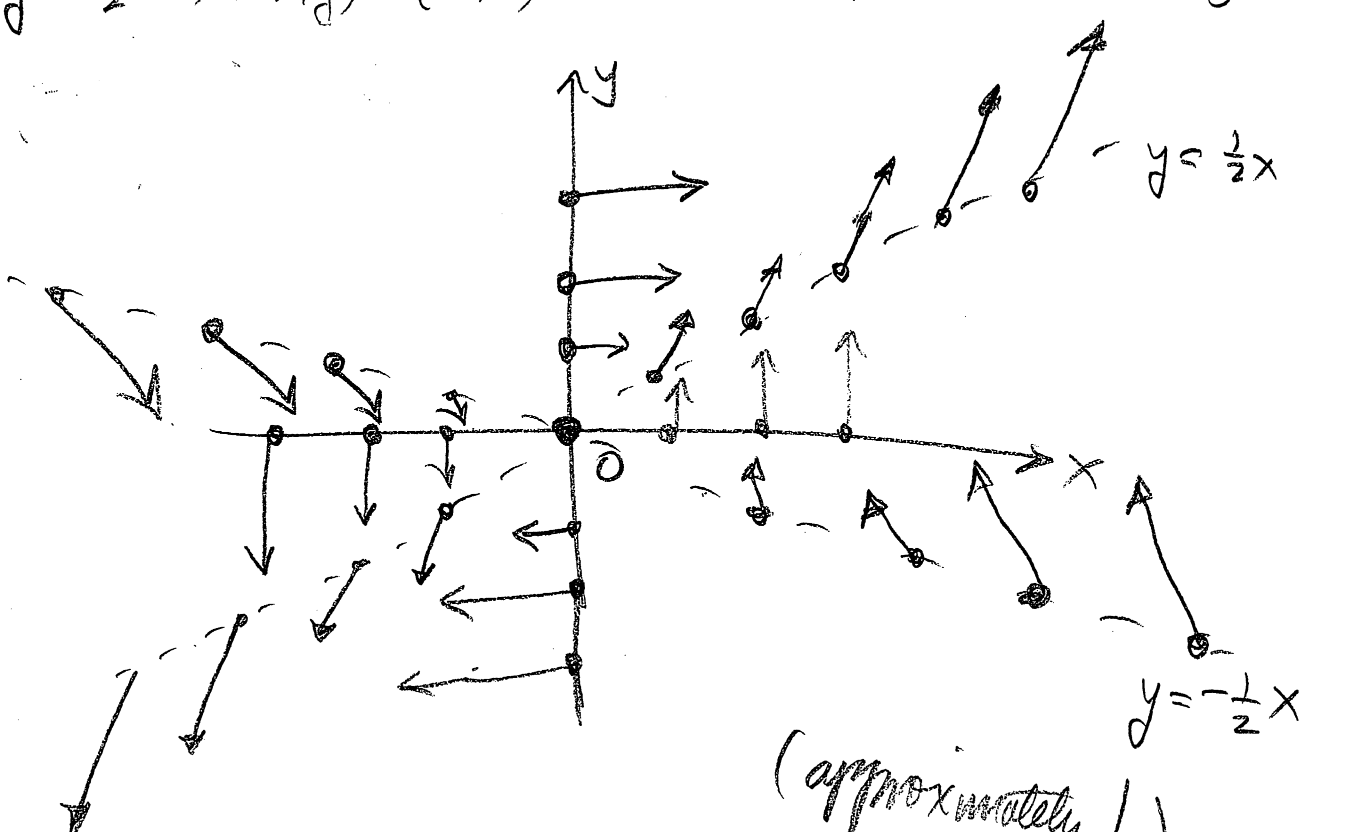
✓ On $y=0 \Rightarrow \vec{F}(x,y) = (0,x)$

✓ On $x=0 \Rightarrow \vec{F}(x,y) = (2y,0)$

✓ On $y=\frac{1}{2}x \Rightarrow \vec{F}(x,y) = (x,x)$

✓ On $y=-\frac{1}{2}x \Rightarrow \vec{F}(x,y) = (-x,x)$

and since $\|\vec{F}\| = \sqrt{4y^2+x^2}$ lengths increase as x, y increase



(approximately!)

Page 243 #9: (a) $V(x,y) = (x,y) \Rightarrow \boxed{(ii)}$ ✓

(b) $V(x,y) = (y,-x) \Rightarrow \boxed{(i)}$ ✓

<u>page 243 #16:</u> $\vec{c}(t) = (\underset{x}{t^2}, \underset{y}{2t-1}, \underset{z}{\sqrt{t}})$

$\vec{F}(x,y,z) = (y+1, 2, \frac{1}{2z})$

<u>Check:</u> $\vec{c}\,'(t) \stackrel{?}{=} \vec{F}(\vec{c}(t))$

$(2t, 2, \frac{1}{2\sqrt{t}}) \stackrel{?}{=} ((2t-1)+1, 2, \frac{1}{2\sqrt{t}})$ ✓

YES. Hence $\vec{c}(t)$ is a flow line of $\vec{F}$

<u>Page 243 #21:</u>

(a) $\vec{F}(x,y,z) = (yz, xz, xy) = \nabla f$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = yz \xrightarrow{I_x} f(x,y,z) = xyz + g(y,z) \\ \frac{\partial f}{\partial y} = xz \xrightarrow{I_y} f(x,y,z) = xyz + h(x,z) \\ \frac{\partial f}{\partial z} = xy \xrightarrow{I_z} f(x,y,z) = xyz + k(x,y) \end{cases} \quad \text{compare forms}$$

$\therefore f(x,y,z) = xyz + C$ ✓

(cont'd)

Page 243 # 21 (b) : $\vec{F}(x,y,z) = (x,y,z) = \nabla f$

$$\Rightarrow \left\{ \begin{array}{l} \frac{\partial f}{\partial x} = x \xrightarrow{I_x} f = \frac{x^2}{2} + g(y,z) \\ \frac{\partial f}{\partial y} = y \xrightarrow{I_y} f = \frac{y^2}{2} + h(x,z) \\ \frac{\partial f}{\partial z} = z \xrightarrow{I_z} f = \frac{z^2}{2} + k(x,y) \end{array} \right\} \text{compare forms}$$

$$\Rightarrow f(x,y,z) = \frac{x^2+y^2+z^2}{2} + C \checkmark$$

4 $\vec{F}(x,y) = \left(\{1+2x\ln y\}, \left\{\frac{x^2}{y}+2y\right\}\right) = \nabla\phi$ 5

Soln 1:

$$\begin{cases} \dfrac{\partial\phi}{\partial x} = 1+2x\ln y \xrightarrow{I_x} \phi(x,y) = x + x^2\ln y + g(y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow D_y \\ \dfrac{\partial\phi}{\partial y} = \dfrac{x^2}{y} + 2y \;=\!=\; \dfrac{\partial\phi}{\partial y} = \dfrac{x^2}{y} + g'(y) \end{cases}$$

$\Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + C$

$\therefore \phi(x,y) = x + x^2\ln y + y^2 + C$

Since $4 = \phi(-1,1) = -1 + 0 + 1 + C \Rightarrow C = 4$

So $\boxed{\phi(x,y) = x + x^2\ln y + y^2 + 4}$ ✓

Soln 2:

$$\left.\begin{cases} \dfrac{\partial\phi}{\partial x} = 1+2x\ln y \xrightarrow{I_x} \phi = x + x^2\ln y + g(y) \\ \dfrac{\partial\phi}{\partial y} = \dfrac{x^2}{y} + 2y \xrightarrow{I_y} \phi = x^2\ln y + y^2 + h(x) \end{cases}\right\} \text{compare forms}$$

$\Rightarrow \phi(x,y) = x + x^2\ln y + y^2 + C$

using $\phi(-1,1) = 4 \Rightarrow \phi(x,y) = x + x^2\ln y + y^2 + 4$

(same as Soln 1)

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page 258 #2: $V(x,y,z) = (yz,\ xz,\ xy)$

$$\Rightarrow \operatorname{div} V = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0+0+0 = 0 \checkmark$$

page 258 #3: $V(x,y,z) = (x,\ y+\cos x,\ z+e^{xy})$

$$\Rightarrow \operatorname{div} V = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y+\cos x) + \frac{\partial}{\partial z}(z+e^{xy}) = 1+1+1 = 3 \checkmark$$

page 258 #14: $\vec{F}(x,y,z) = (yz,\ xz,\ xy)$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (yz) & (xz) & (xy) \end{vmatrix} = \left(\{x-x\},\ -\{y-y\},\ \{z-z\}\right)$$

$$= (0,0,0) = \vec{0} \checkmark$$

page 258 #24: ($f: \mathbb{R}^3 \to \mathbb{R}$ is C^2 fcn)

(a) curl(grad f) ← O.K.

(b) grad(curl f) ← DNE (cannot take curl of a function)

(c) div(grad f) ← O.K.

(d) grad(div f) ← DNE (cannot take div of a function)

(e) curl(div f) ← DNE (cannot take div of a function)

(f) div(curl f) ← O.K.

$\boxed{6}$ Prove $\underline{\text{curl}(\nabla f) = \vec{0}}$ i.e, $\nabla \times (\nabla f) = \vec{0}$:

Since $f: \mathbb{R}^3 \to \mathbb{R} \Rightarrow \nabla f = (f_x, f_y, f_z)$

Now $\text{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$

$= \left(\{f_{zy} - f_{yz}\}, -\{f_{zx} - f_{xz}\}, \{f_{yx} - f_{xy}\} \right)$

Now since $f \in C^2$, the mixed partial derivatives are equal

$= (0, 0, 0) = \vec{0}$