SOLUTIONS

MA 362 - Fall 2016

Homework Set # 9 (updated)

1. (§5.3) **Page 288:** # 3(c)(d), 9.
2. (§5.4) **Page 293:** # 2, 4(d), 9.
3. (§5.5) **Page 303:** # 10, 18.
4. Let W be that part of the solid sphere $x^2+y^2+z^2 \leq 25$ which lies above the plane $z=3$. Set up but do not evaluate the triple integral

$$\iiint_W (2x^2+2y^2+7z)\,dV$$

in *Rectangular, Cylindrical and Spherical Coordinates.*

5. Fill in the boxes: $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3}(5x-4z)\,dz\,dy\,dx = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square}\boxed{}\,dy\,dx\,dz$
6. (§6.1) **Page 313:** # 7.
7. (§6.2) **Page 326:** # 3, 5(b), 19.
8. (§6.3) **Page 337:** # 5.

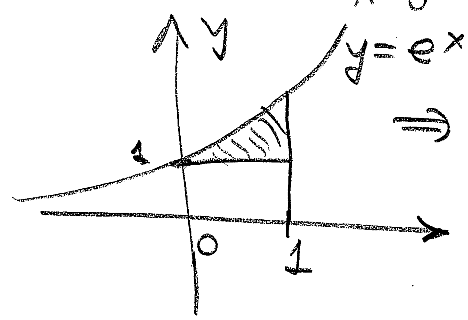
Solutions

$\boxed{1}$ <u>page 288 # 3(c)</u>: $\displaystyle\int_0^1\int_1^{e^x}(x+y)\,dy\,dx = \int_0^1\left(xy+\frac{y^2}{2}\right)\Bigg|_{y=1}^{e^x}dx$

$$= \int_0^1 \left(xe^x+\frac{e^{2x}}{2}\right)-\left(x+\frac{1}{2}\right)dx$$

$$= \left(xe^x-e^x+\frac{e^{2x}}{4}-\frac{x^2}{2}-\frac{x}{2}\right)\Bigg|_{x=0}^{1} = \left(\frac{e^2}{4}-\frac{1}{2}-\frac{1}{2}\right)-\left(-1+\frac{1}{4}\right)$$

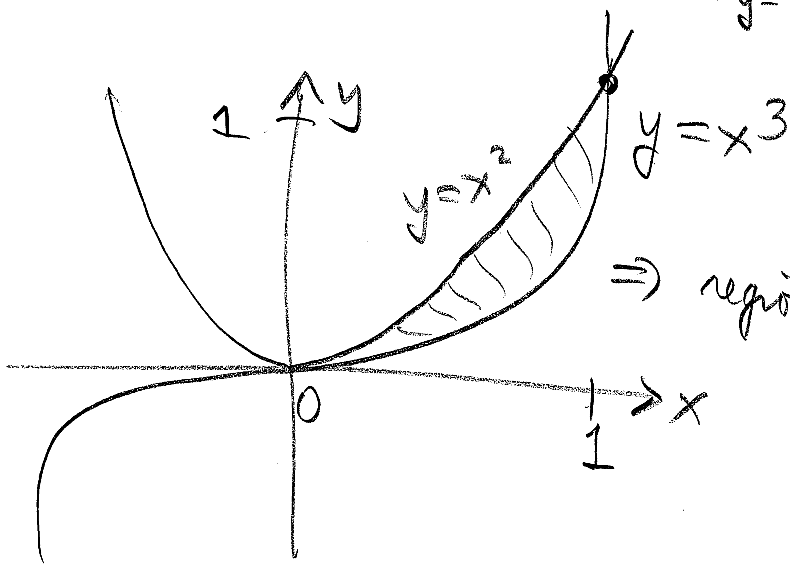
$$= \frac{e^2-1}{4} \checkmark$$



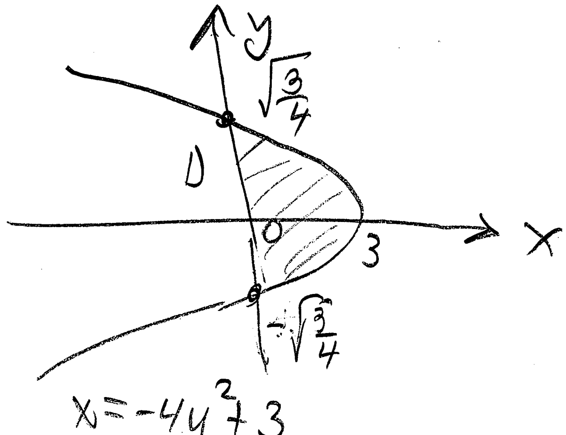
$\Rightarrow$ region is both x and y simple

<u>page 288 # 3(d)</u>: $\displaystyle\int_0^1\int_{x^3}^{x^2} y\,dy\,dx = \int_0^1 \frac{y^2}{2}\Bigg|_{y=x^3}^{x^2}dx = \int_0^1\left(\frac{x^4}{2}-\frac{x^6}{2}\right)dx$

$$= \frac{1}{35} \checkmark$$



$\Rightarrow$ region is both x, y simple page 288 # 9:



$$D: \begin{cases} 0 \le x \le -4y^2+3 \\ -\sqrt{\frac{3}{4}} \le y \le \sqrt{\frac{3}{4}} \end{cases}$$

$$\therefore \iint_D x^3 y \, dx\, dy = \int_{-\sqrt{\frac{3}{4}}}^{\sqrt{\frac{3}{4}}} \left(\int_0^{-4y^2+3} x^3 y \, dx \right) dy = 0 \checkmark$$

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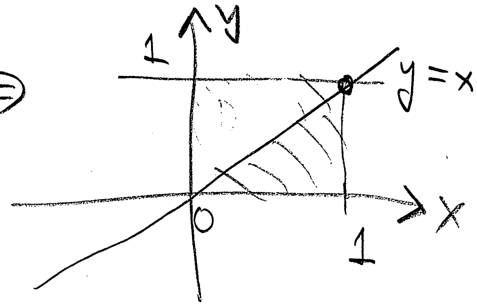
2 page 293 # 2: $I = \int_0^1 \int_y^1 \sin(x^2)\, dx\, dy \Rightarrow D: \begin{cases} y \le x \le 1 \\ 0 \le y \le 1 \end{cases}$ x-simple

$\Rightarrow$



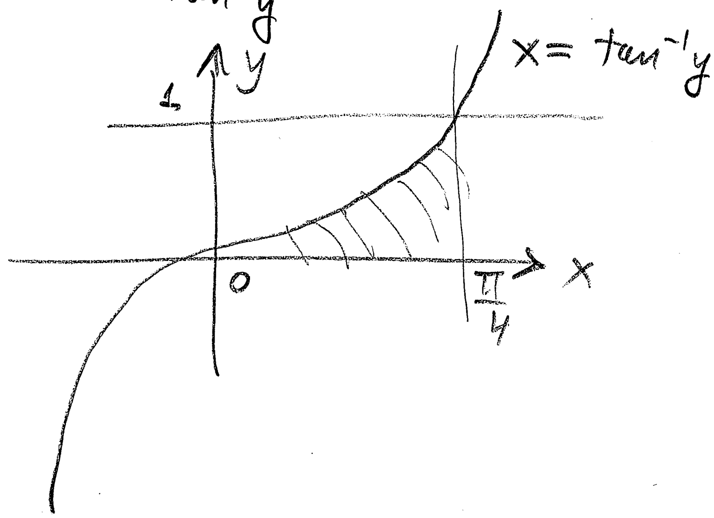
D also y-simple : $D: \begin{cases} 0 \le y \le x \\ 0 \le x \le 1 \end{cases}$

$$\therefore I = \int_0^1 \int_0^x \sin(x^2)\, dy\, dx = \int_0^1 x \sin(x^2)\, dx$$

$$= \frac{1-\cos 1}{2} \checkmark$$

page 293 # 4(d): $I = \int_0^1 \int_{\tan^{-1} y}^{\frac{\pi}{4}} \sec^5 x \, dx \, dy$

$$D: \begin{cases} \tan^{-1} y \le x \le \frac{\pi}{4} \\ 0 \le y \le 1 \end{cases} \Rightarrow$$



D also y-simple: $D: \begin{cases} 0 \le y \le \tan x \\ 0 \le x \le \frac{\pi}{4} \end{cases}$

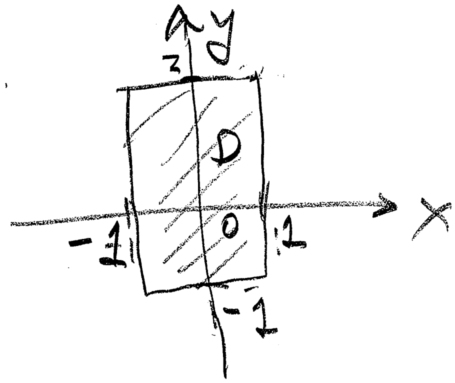
$$\therefore I = \int_0^{\frac{\pi}{4}} \int_0^{\tan x} \sec^5 x \, dy \, dx = \int_0^{\frac{\pi}{4}} \sec^5 x \tan x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^4 x (\sec x \tan x) dx = \int_1^{\frac{2}{\sqrt{2}}} u^4 \, du = \frac{2^{5/2} - 1}{5} \checkmark$$

let $u = \sec x$

$\therefore du = \sec x \tan x \, dx$

page 293 #9:



$A(D) = 6$

Now $\frac{1}{6} = \frac{1}{1^2+2^2+1} \le \frac{1}{x^2+y^2+1} \le \frac{1}{0+1} = 1$

$$\therefore \underbrace{\iint_D \frac{1}{6}\, dA}_{\frac{1}{6}(6)} \le \iint_D \frac{1}{x^2+y^2+1}\, dA \le \underbrace{\iint_D 1\, dA}_{1(6)}$$

$$\therefore 1 \le \iint_D \frac{1}{x^2+y^2+1}\, dA \le 6$$

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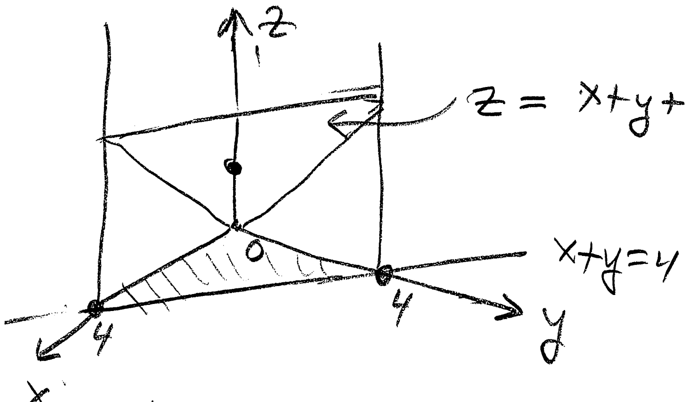
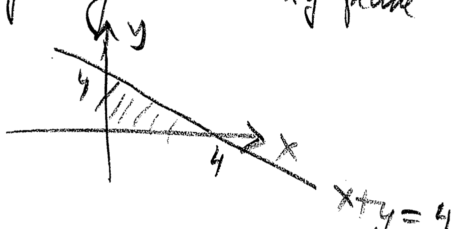
3 page 303 #10: W is bounded by planes $x=0, y=0, z=0,$ $x+y=4,\ x = z-y-1$



$$\therefore W: \begin{cases} 0 \le z \le x+y+1 \\ 0 \le y \le 4-x \\ 0 \le x \le 4 \end{cases}$$

Projecting W onto xy plane is

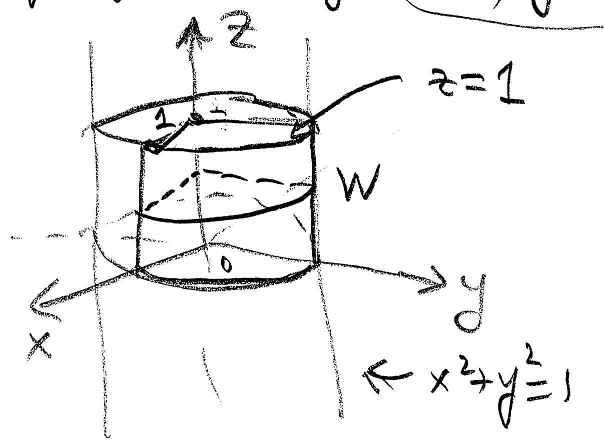


$$\text{or } W: \begin{cases} 0 \le z \le x+y+1 \\ 0 \le x \le 4-y \\ 0 \le y \le 4 \end{cases}$$

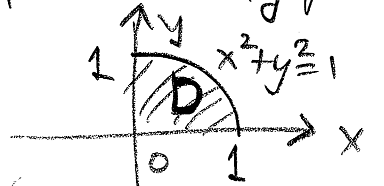
page 303 #18 : $I = \iiint_W z \, dx\, dy\, dz$

W bounded by $x=0,\ y=0,\ z=0,\ z=1,\ x^2+y^2=1$ with $x, y \geq 0$

$\therefore$ W lies in 1^{st} octant



Soln 1: (Project W onto xy plane) $W: \begin{cases} 0 \leq z \leq 1 \\ 0 \leq y \leq \sqrt{1-x^2} \\ 0 \leq x \leq 1 \end{cases}$



$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\int_0^1 z\, dz \right) dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2}\, dy\, dz$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dz = \frac{1}{2}\,(\text{Area of } D) = \frac{1}{2}\left\{ \left(\frac{1}{4}\right) \pi (1)^2 \right\}$$

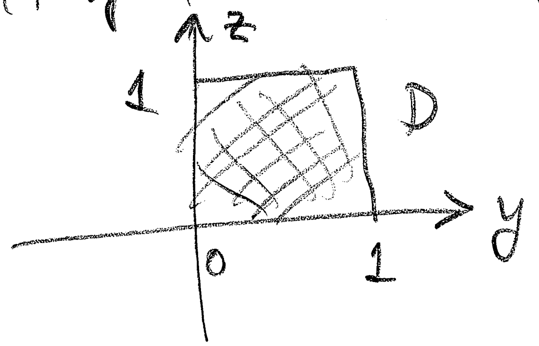
$$= \boxed{\frac{\pi}{8}} \checkmark$$

Soln 2: $\boxed{CC}$ $W: \begin{cases} 0 \leq z \leq 1 \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 z\, \underset{\uparrow \text{ Jacobian}}{[r]} \cdot dz\, dr\, d\theta = \frac{\pi}{8} \checkmark$$

(cont'd)

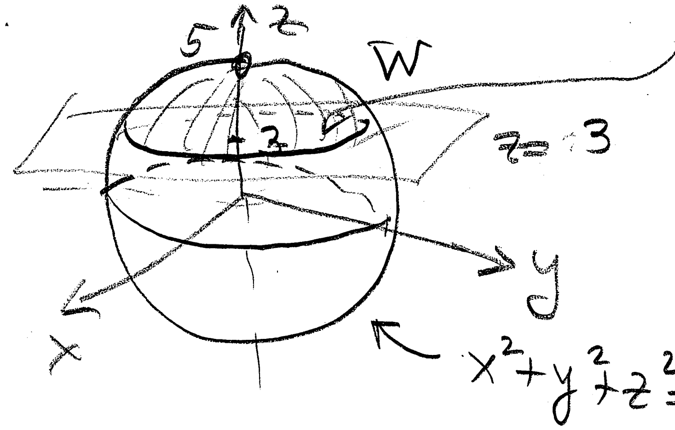
Soln 3: (Project W onto yz plane)



So $W: \begin{cases} 0 \le x \le \sqrt{1-y^2} \\ 0 \le y \le 1 \\ 0 \le z \le 1 \end{cases}$

$$\therefore I = \int_0^1 \left(\int_0^1 \left(\int_0^{\sqrt{1-y^2}} z \, dx \right) dy \right) dz = \frac{\pi}{8} \checkmark$$

$\boxed{4}$

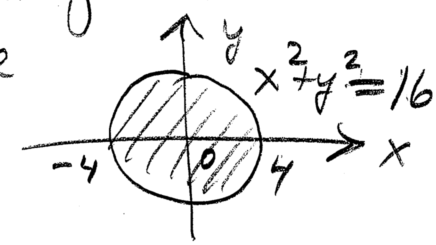


intersection is $x^2+y^2+3^2=25$

$x^2+y^2=16$

$\boxed{RC}$ $W: \begin{cases} 3 \le z \le \sqrt{25-x^2-y^2} \\ -\sqrt{16-x^2} \le y \le \sqrt{16-x^2} \\ -4 \le x \le 4 \end{cases}$

projection of W onto xy plane



$$\Rightarrow I = \iiint_W (2x^2+2y^2+7z)\, dV$$

$$= \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} (2x^2+2y^2+7z)\, dz\, dy\, dx \checkmark$$

$\boxed{CC}$ $W: \begin{cases} 3 \le z \le \sqrt{25-r^2} \\ 0 \le r \le 4 \\ 0 \le \theta \le 2\pi \end{cases}$

since $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

$$\therefore I = \iiint_W (2x^2+2y^2+7z)\, dV = \int_0^{2\pi} \int_0^4 \int_3^{\sqrt{25-r^2}} (2r^2+7z)\, \underbrace{[r]}_{\text{Jacobian}}\, dz\, dr\, d\theta \checkmark$$

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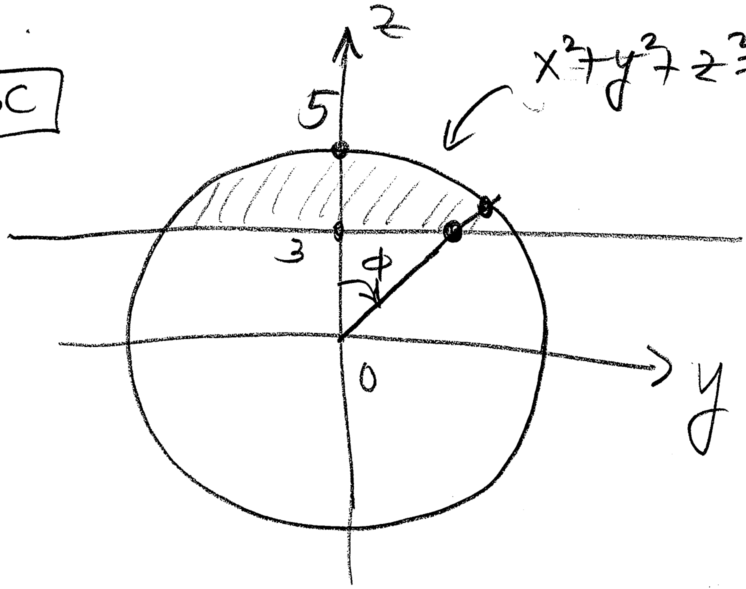
SC



$x^2+y^2+z^2=25$ or $\rho=5$

$z=3$ or $\rho\cos\phi=3$

$$\rho=\frac{3}{\cos\phi}$$

max ϕ occurs when $\rho=5$ and $\rho=\frac{3}{\cos\phi}$ intersect

$$\Rightarrow\ 5=\frac{3}{\cos\phi}$$

$$\phi=\cos^{-1}\left(\tfrac{3}{5}\right)$$

$$\therefore\ W:\begin{cases}\dfrac{3}{\cos\phi}\le\rho\le 5\\[4pt] 0\le\phi\le\cos^{-1}\left(\tfrac{3}{5}\right)\ {}^{*}\\[4pt] 0\le\theta\le 2\pi\end{cases}$$

Jacobian $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\rho^2\sin\phi$ since $\begin{cases}x=(\rho\sin\phi)\cos\theta\\ y=(\rho\sin\phi)\sin\theta\\ z=\rho\cos\phi\end{cases}$

$$\therefore\ I=\iiint_W\left(2x^2+2y^2+7z\right)dV$$

$$=\int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{3}{5}\right)}\int_{\frac{3}{\cos\phi}}^{5}\left(2\rho^2\sin^2\phi+7\rho\cos\phi\right)\left[\rho^2\sin\phi\right]d\rho\,d\phi\,d\theta$$

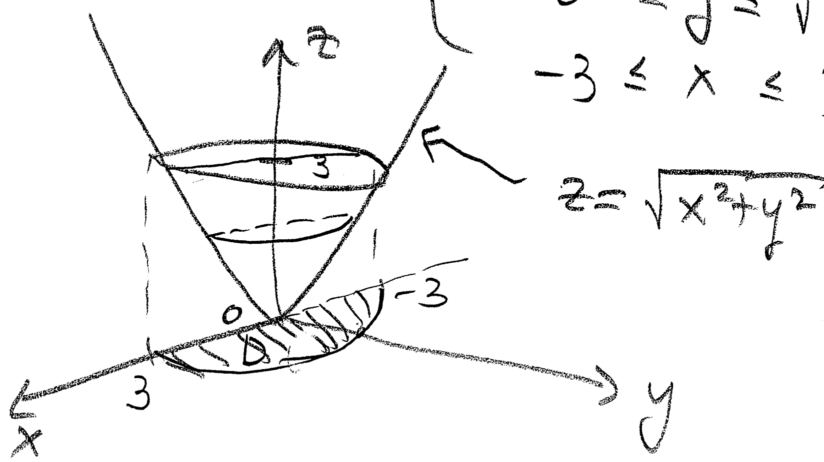
Jacobian

* **Note:** $\cos^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{4}{3}\right)$

$\boxed{5}$ $I = \displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3} (5x-4z)\, dz\, dy\, dx = \int_{\Box}^{\Box}\int_{\Box}^{\Box}\int_{\Box}^{\Box} \Box \, dy\, dx\, dz$

Now the solid W is $\begin{cases} \sqrt{x^2+y^2} \le z \le 3 \\ 0 \le y \le \sqrt{9-x^2} \\ -3 \le x \le 3 \end{cases}$

$\Rightarrow$



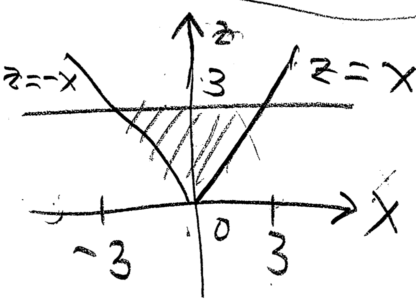
$\therefore$ W is half of the cone below $z = 3$.

The integral wants us to integrate in this order y, x, z

$\therefore$ want W to be bounded by surfaces $y = y_1(x, z)$ and $y = y_2(x, z)$

and project W onto xz plane.

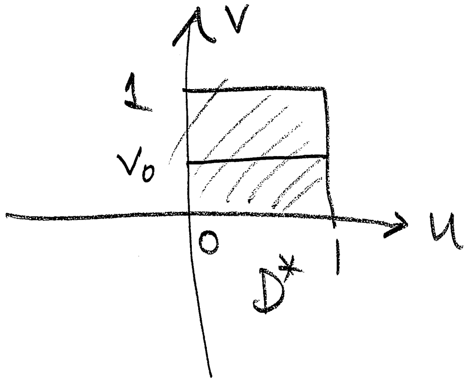
Thus W: $\begin{cases} 0 \le y \le \sqrt{z^2 - x^2} \\ -z \le x \le z \\ 0 \le z \le 3 \end{cases}$



$\therefore$ $I = \displaystyle\int_{\boxed{0}}^{\boxed{3}}\int_{\boxed{-z}}^{\boxed{z}}\int_{\boxed{0}}^{\boxed{\sqrt{z^2-x^2}}} \boxed{5x-4z}\, dy\, dx\, dz$ ✓

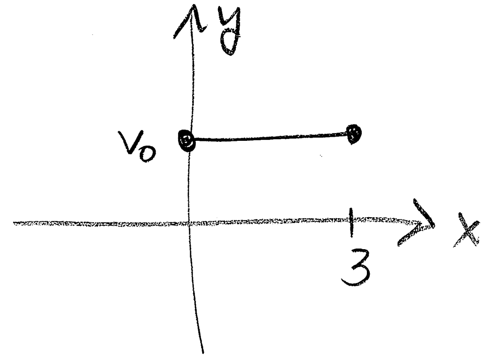
[6] page 313 #7: $T(u,v) = (-u^2+4u, v)$

(10)



$\xrightarrow{\hspace{2em}}$

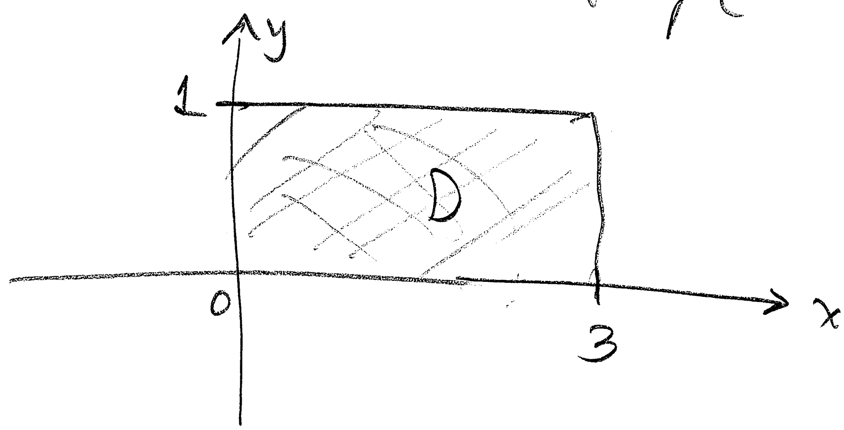
$T: \begin{cases} x = -u^2+4u \\ y = v \end{cases}$



Consider image of horizontal lines $v = v_0$: Then $\begin{cases} x = -u^2+4u \\ y = v_0 \end{cases}$

is a line $y = v_0$
with $0 \le x \le 3$
(since $0 \le u \le 1$)

Thus since $0 \le v_0 \le 1$, D must be the rectangle



$D = [0,3] \times [0,1]$

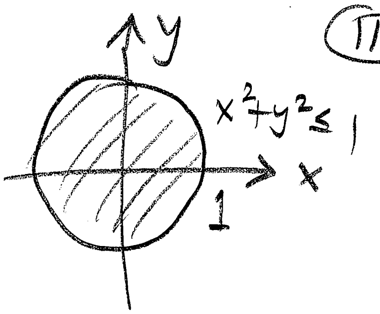
T one-to-one ? Suppose $T(u_1,v_1) = T(u_2,v_2)$ then

$(-u_1^2+4u_1, v_1) = (-u_2^2+4u_2, v_2)$

$\Rightarrow \begin{cases} -u_1^2+4u_1 = -u_2^2+4u_2 \\ v_1 = v_2 \end{cases}$ This system only true if $(u_1,v_1) = (u_2,v_2)$

Hence T is indeed 1-1

7 page 326 #3: $I = \iint_D e^{(x^2+y^2)} dx\,dy$ (11)



Change of Variables for PC $\Rightarrow \dfrac{\partial(x,y)}{\partial(r,\theta)} = r$

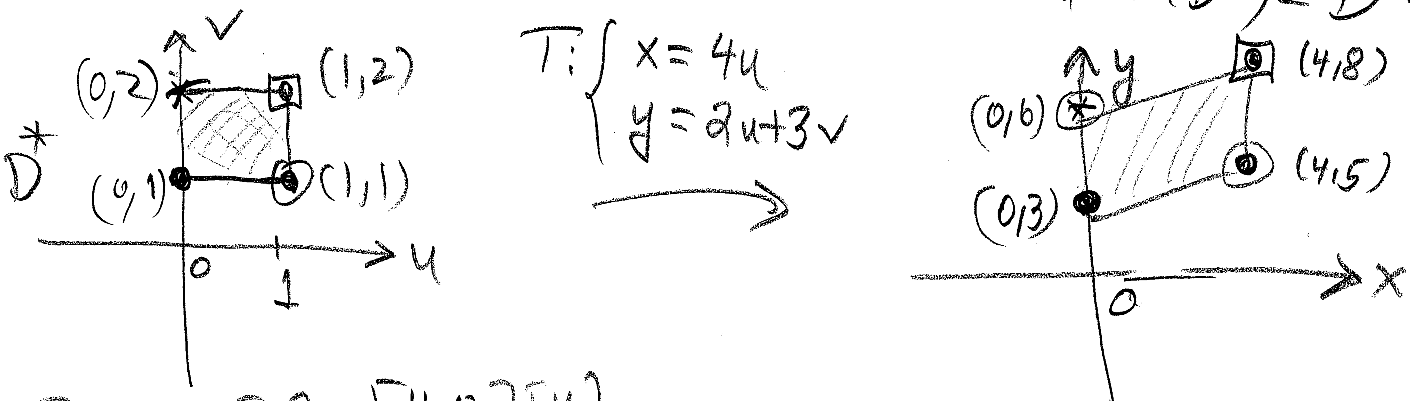
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\therefore\ I = \int_0^{2\pi} \left(\int_0^1 e^{r^2} [r]\, dr \right) d\theta = \frac{\pi}{2}(e-1) \checkmark$$

Jacobian

page 326 #5(b): $I = \iint_D (x-y)\,dx\,dy$ where

$T(u,v) = (4u, 2u+3v)$, $D^* = [0,1] \times [1,2]$ and $T(D^*) = D$:



Since $T\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$ is a linear map, T maps parallelograms to parallelograms and vertices to vertices. Thus look at vertices of D^* and see where they are mapped to:

(u,v)	$\mapsto$	(x,y)
$(0,1)$	$\mapsto$	$(0,3)$
$(1,1)$	$\mapsto$	$(4,5)$
$(1,2)$	$\mapsto$	$(4,8)$
$(0,2)$	$\mapsto$	$(0,6)$

See picture above, note that D is a parallelogram and it is y-simple

Hence,

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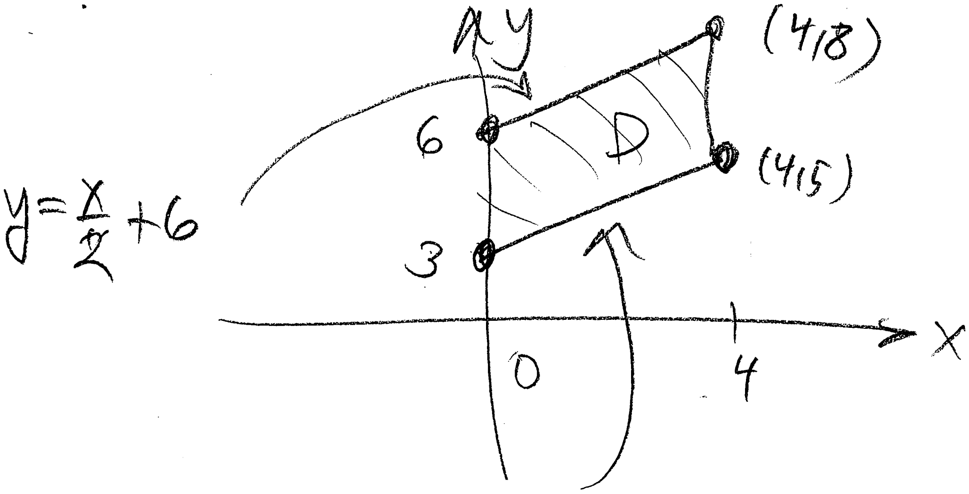


$$y = \frac{x}{2} + 3$$

$\therefore \quad D : \begin{cases} \frac{x}{2} + 3 \le y \le \frac{x}{2} + 6 \\ 0 \le x \le 4 \end{cases}$

Since $T(u,v) = (\underbrace{4u}_{x}, \underbrace{2u+3v}_{y}) \Rightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = 12$

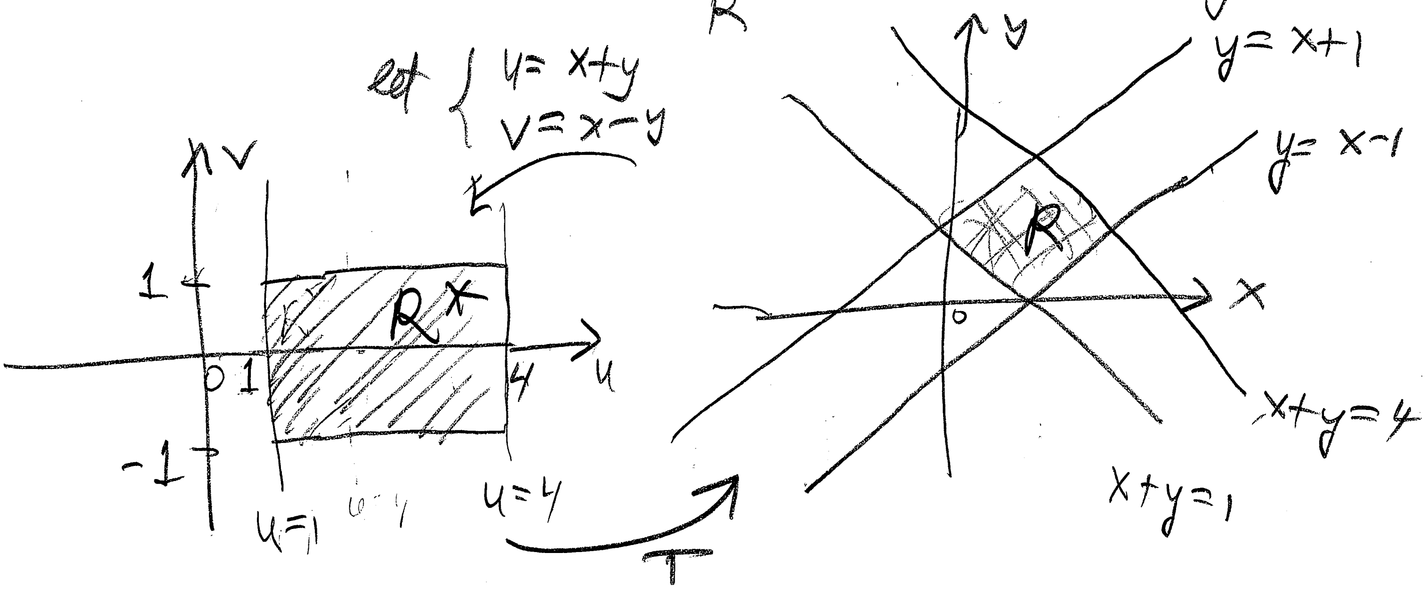
$$\therefore \iint_D (x-y)\,dx\,dy = \iint_{D^*} \{4u - (2u+3v)\} \left[|12| \right] du\,dv$$

Jacobian

$$= \int_1^2 \int_0^1 12(2u-3v)\,du\,dv = -42$$

page 326 #19 : $I = \iint\limits_R (x+y)^2 e^{(x-y)}\, dx\,dy$

let $\begin{cases} u = x+y \\ v = x-y \end{cases}$



If $\begin{cases} u = x+y \\ v = x-y \end{cases}$ $\Longrightarrow$ the image of lines $x+y=1$ and $x+y=4$ are lines $u=1$ and $u=4$ in (u,v) plane

the image of lines $y=x-1$ and $y=x+1$

i.e., $x-y=1$ and $x-y=-1$

are mapped to $v=1$ and $v=-1$ respectively

Thus R^* is the rectangle in (u,v) plane as shown above.

The mapping ⊛ $\begin{cases} u = x+y \\ v = x-y \end{cases}$ is T^{-1}, not T.

To find T, solve for x, y in terms of u, v in ⊛

$\therefore\ T: \begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u - \frac{1}{2}v \end{cases}$

(cont'd)

Now $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$

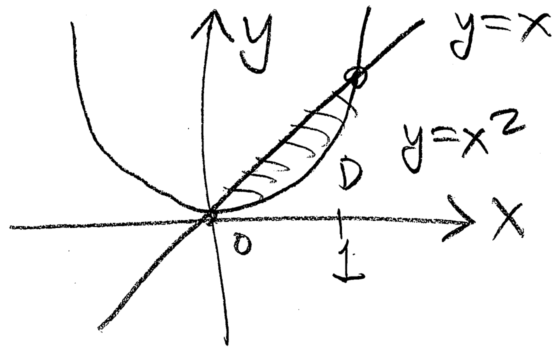
$\therefore$ $I = \iint\limits_R (x+y)^2 e^{(x-y)}\, dx\, dy = \iint\limits_{R^*} u^2 e^v \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$

$= \displaystyle\int_{-1}^{1} \left(\int_1^4 u^2 e^v \left(\tfrac{1}{2}\right) du \right) dv$

$= \dfrac{4^3 - 1}{6}\left(e - e^{-1}\right)$

$= \dfrac{21}{2}\left(e - e^{-1}\right)$ ✓

$\boxed{8}$ page 337 #5



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$\rho(x,y) = x+y$

$$\therefore \quad \bar{x} = \frac{M_y}{M} = \frac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA} = \frac{\int_0^1 \int_{x^2}^{x} x(x+y)\,dy\,dx}{\int_0^1 \int_{x^2}^{x} (x+y)\,dy\,dx}$$

$$\Rightarrow \quad \bar{x} = \frac{11/120}{3/20} = \frac{11}{18}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA} = \frac{\int_0^1 \int_{x^2}^{x} y(x+y)\,dy\,dx}{\int_0^1 \int_{x^2}^{x} (x+y)\,dy\,dx}$$

$$\Rightarrow \quad \bar{y} = \frac{13/168}{3/20} = \frac{65}{126}$$

$$\therefore \quad (\bar{x}, \bar{y}) = \frac{65}{126} \quad \checkmark$$