SOLUTIONS

MA 362 - Fall 2016

Homework Set # 10

1. (§7.1) **Page 356:** # 10.

2. (§7.2) **Page 373:** # 3(d), 4.

3. If C is the curve $y = x^2 + 4$ from $(0,4)$ to $(2,8)$, compute the following:

   (a) Path Integral: $I = \int_C f\,ds$, where $f(x,y) = xy - x^3$.

   (b) Line Integral: $J = \int_C \tilde{\mathbf{F}} \bullet d\tilde{\mathbf{s}}$, where $\tilde{\mathbf{F}}(x,y) = (1+xy)\,\mathbf{i} - x\,\mathbf{j}$.

   (c) Line Integral (in differential form): $K = \int_C 2xy\,dx + x\,dy$.

4. Evaluate the line integral $\int_C \tilde{\mathbf{F}} \bullet d\tilde{\mathbf{s}}$, where $\tilde{\mathbf{F}}(x,y) = \left(2xy^3,\ \{3x^2y^2 - 4y\}\right)$ and C is any smooth curve starting at $(1,0)$ and ending at $(2,2)$.

   Hint: $\tilde{\mathbf{F}}$ is a gradient field $\tilde{\mathbf{F}}(x,y) = \nabla f(x,y)$.

5. (§7.3) **Page 381:** # 1.

6. Do the following problems from the book:

   (a) (§4.4) **Page 259:** # 23, 25.
   (b) (§5.3) **Page 288:** # 1, 9.
   (c) (§5.4) **Page 293:** # 5.
   (d) (§5.5) **Page 302:** # 1.

[1] page 356 #10 : $\vec{c}(t) = (\sin t, \cos t, t)$, $0 \le t \le 2\pi$

$x'' \quad y'' \quad z''$

$ds = \|\vec{c}\,'(t)\|\,dt = \sqrt{\cos^2 t + \sin^2 t + 1^2}\,dt = \sqrt{2}\,dt$ ✓

ⓐ $\displaystyle\int_{\vec{c}} (x+y+z)\,ds = \int_0^{2\pi} (\sin t + \cos t + t)\sqrt{2}\,dt = \underline{\underline{2\pi^2\sqrt{2}}}$

ⓑ $\displaystyle\int_{\vec{c}} \cos z\,ds = \int_0^{2\pi} \cos t\ \sqrt{2}\,dt = \underline{\underline{0}}$

[2] page 373 #3(d): $\vec{c}(t) = (t^2, 3t, 2t^3)$, $-1 \le t \le 2$

$x'' \quad y'' \quad z''$

$\vec{F}(x,y,z) = (x,y,z)$, $\quad d\vec{s} = \vec{c}\,'(t)\,dt = (2t, 3, 6t^2)\,dt$ ✓

$\displaystyle\therefore \int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_{-1}^{2} \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt = \int_{-1}^{2} (t^2, 3t, 2t^3)\cdot(2t, 3, 6t^2)\,dt$

$\displaystyle = \int_{-1}^{2} \{2t^3 + 9t + 12t^5\}\,dt = \underline{147}$

page 373 #4

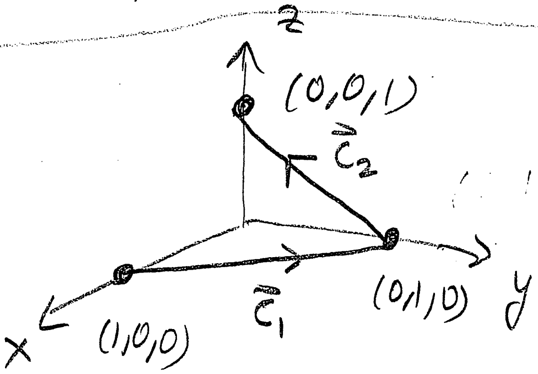
(a) $I=\int_{\vec{c}} x\,dy - y\,dx$, where $\vec{c}(t)=(\underset{x}{\underset{\shortparallel}{\cos t}}, \underset{y}{\underset{\shortparallel}{\sin t}})$, $0\le t\le 2\pi$

$$\therefore\ I=\int_0^{2\pi}(\cos t)(\cos t)\,dt-(\sin t)(-\sin t)\,dt=\int_0^{2\pi}dt=\underline{\underline{2\pi}}$$

(b) $I=\int_{\vec{c}} x\,dx+y\,dy$, where $\vec{c}(t)=(\underset{x}{\underset{\shortparallel}{\cos\pi t}}, \underset{y}{\underset{\shortparallel}{\sin\pi t}})$, $0\le t\le 2$

$$\therefore\ I=\int_0^{2}(\cos\pi t)(-\pi\sin\pi t)\,dt+(\sin\pi t)(\pi\cos\pi t)\,dt=\underline{\underline{0}}$$

(c) $I=\int_{\vec{c}} yz\,dx+xz\,dy+xy\,dz$



$\vec{c}$ is union of two line segments $\vec{c}_1$ and $\vec{c}_2$

Recall, line segment from P to Q is $\vec{r}(t)=P+t\,\vec{PQ}$, $0\le t\le 1$

$$\therefore\ \vec{c}_1(t)=(1,0,0)+t(-1,1,0)=(\overset{=x}{1-t},\overset{=y}{t},\overset{=z}{0}),\quad 0\le t\le 1$$

and $$\vec{c}_2(t)=(0,1,0)+t(0,-1,1)=(\underset{x}{\underset{\shortparallel}{0}},\underset{y}{\underset{\shortparallel}{1-t}},\underset{z}{\underset{\shortparallel}{t}}),\quad 0\le t\le 1$$

$$\therefore\ I=\int_{\vec{c}_1} yz\,dx+xz\,dy+xy\,dz+\int_{\vec{c}_2} yz\,dx+xz\,dy+xy\,dz$$

$$I=\int_0^1(0+0+0)\,dt+\int_0^1(0+0+0)\,dt=\underline{\underline{0}}$$

page 373 #4(d)

$I = \int_{c} x^2 dx - xy\, dy + dz$ where $z = x^2$, $y = 0$
from $(-1,0,1)$ to $(1,0,1)$

$\therefore \vec{c}(t) = (\underset{x}{\overset{\parallel}{t}}, \underset{y}{\overset{\parallel}{0}}, \underset{z}{\overset{\parallel}{t^2}})$, $-1 \le t \le 1$

Hence $I = \int_{-1}^{1} \{ t^2 dt - t(0)(0) + 2t\, dt \}$

$I = \int_{-1}^{1} (t^2 + 2t)\, dt = \underline{\underline{\frac{2}{3}}}$

$\boxed{3}$ $C: y = x^2 + 4$ from $(0,4)$ to $(2,8)$

So a parametrization of C is $\vec{c}(t) = (\underset{x}{\underset{\shortparallel}{t}}, \underset{y}{\underset{\shortparallel}{t^2+4}})$, $0 \le t \le 2$

ⓐ $I = \int_C f\, ds = \int_0^2 f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\, dt$, since $f(x,y) = xy - x^3$

$\Rightarrow I = \int_0^2 \{t(t^2+4) - t^3\} \sqrt{1+4t^2}\, dt = \int_0^2 4t\sqrt{1+4t^2}\, dt$

$\therefore$ $I = \underline{\frac{1}{3}(17^{3/2} - 1)}$

ⓑ $J = \int_C \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$, since $\vec{F}(x,y) = (1+xy, -x)$

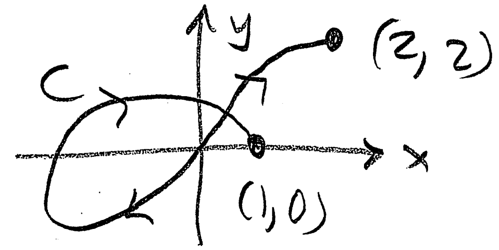
$\Rightarrow J = \int_0^2 (1 + t(t^2+4), -t) \cdot (1, 2t)\, dt = \int_0^2 (1 + t^3 + 4t - 2t^2)\, dt$

$\therefore$ $J = \underline{26/3}$

ⓒ $K = \int_C 2xy\, dx + x\, dy = \int_0^2 2(t)(t^2+4)\, dt + (t)(2t\, dt)$

$= \int_0^2 (2t^3 + 8t + 2t^2)\, dt = \underline{88/3}$

4 $I = \int_C \vec{F} \cdot d\vec{s}$, where



$\vec{F}(x,y) = (2xy^3,\ 3x^2y^2 - 4y) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. Find $f(x,y)$:

Soln 1: (MA 366 method)

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy^3 \xrightarrow{\ I_x\ } f(x,y) = x^2y^3 + h(y) \\ \qquad\qquad\qquad\qquad\qquad \downarrow D_y \\ \frac{\partial f}{\partial y} = 3x^2y^2 - 4y \ = \ \frac{\partial f}{\partial y} = 3x^2y^2 + h'(y) \end{cases}$$

$\Rightarrow h'(y) = -4y$

$\therefore h(y) = -2y^2 + C$

$\therefore$ ~~Potential~~ fcn is $f(x,y) = x^2y^3 - 2y^2 + C$

Soln 2: (Student method)

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy^3 \xrightarrow{\ I_x\ } f(x,y) = x^2y^3 + h(y) \\ \frac{\partial f}{\partial y} = 3x^2y^2 - 4y \xrightarrow{\ I_y\ } f(x,y) = x^2y^3 - 2y^2 + g(x) \end{cases}$$

compare forms

$\therefore f(x,y) = x^2y^3 - 2y^2 + C$, same as above.

Finally, $\int_C \vec{F} \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} = f(2,2) - f(1,0) = \underline{24}$

FTC for Gradient fields

5 page 381 #1 : $\Phi(u,v) = (\underset{x}{\underset{\parallel}{2u}}, \underset{y}{\underset{\parallel}{u^2+v}}, \underset{z}{\underset{\parallel}{v^3}})$ (6)

normal vector to surface is $\vec{n} = \Phi_u \times \Phi_v$

i.e. $\vec{n} = (2, 2u, 0) \times (0, 1, 2v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2u & 0 \\ 0 & 1 & 2v \end{vmatrix} = (4uv, -4v, 2)$

when $(x_0, y_0, z_0) = (0,1,1) \Rightarrow \begin{array}{l} u_0 = 0 \\ v_0 = 1 \end{array}$

$\therefore$ normal vector @ $(x_0, y_0, z_0) = (0,1,1)$ is $\vec{n} = (0, -4, 2)$

$\therefore$ Eqn of ~~tangent~~ plane is

$(x-0, y-1, z-1) \cdot (0, -4, 2) = 0$

i.e. $\boxed{-4(y-1) + 2(z-1) = 0}$ ✓

Book Problems

page 259 #23: $\vec{F}(x,y,z) = (e^{xz}, \sin(xy), x^5y^3z^2)$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial(e^{xz})}{\partial x} + \frac{\partial(\sin(xy))}{\partial y} + \frac{\partial(x^5y^3z^2)}{\partial z}$$

$$= ze^{xz} + x\cos(xy) + 2x^5y^3z \checkmark$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xz} & \sin(xy) & (x^5y^3z^2) \end{vmatrix}$$

$$= \left(\{3x^5y^2z^2\}, -\{5x^4y^3z^2 - xe^{xz}\}, \{y\cos(xy)\}\right) \checkmark$$

page 259 #25

(a) curl(grad $\vec{F}$) ← Nonsense (can't take grad of vector field)

(b) grad(curl $\vec{F}$) ← Nonsense (can only take grad of a function)

(c) div(grad $\vec{F}$) ← Nonsense (can't take grad of vector field)

(d) grad(div $\vec{F}$) ← Meaniful [this is a vector field]

(e) curl(div $\vec{F}$) ← Nonsense (can't take curl of a function)

(f) div(curl $\vec{F}$) ← Meaniful [this is a scalar function]

Page 288 #1

(a) $\int_1^2 \int_{\ln x}^{e^x} dy\,dx = \iint_D dy\,dx$, where $D: \begin{cases} \ln x \le y \le e^x \\ 1 \le x \le 2 \end{cases}$ $\therefore$ $\boxed{(iii)}$ ✓

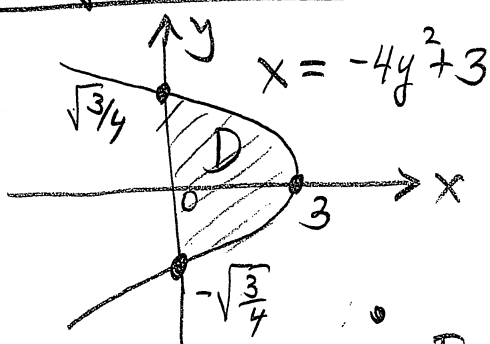
(b) $\int_0^2 \int_{\frac{x}{8}}^{x^{1/3}} dy\,dx = \iint_D dy\,dx$, where $D: \begin{cases} \frac{x}{8} \le y \le x^{1/3} \\ 0 \le x \le 2 \end{cases}$ $\therefore$ $\boxed{(iv)}$ ✓

(c) $\int_0^2 \int_{-\sqrt{9-y^2}}^{0} dx\,dy = \iint_D dx\,dy$, where $D: \begin{cases} -\sqrt{9-y^2} \le x \le 0 \\ 0 \le y \le 2 \end{cases}$ $\therefore$ $\boxed{(iii)}$ ✓

(d) $\int_0^3 \int_{\cos^{-1}\left(\frac{y}{3}\right)}^{0} dx\,dy = \iint_D dx\,dy$, where $D: \begin{cases} \cos^{-1}\left(\frac{y}{3}\right) \le x \le 0 \\ 0 \le y \le 3 \end{cases}$ $\therefore$ $\boxed{(i)}$ ✓

Page 288 #9:

$I = \iint_D x^3 y\,dx\,dy$, where D is as shown.



Since D is x-simple $\Rightarrow D: \begin{cases} 0 \le x \le -4y^2+3 \\ -\sqrt{\frac{3}{4}} \le y \le \sqrt{\frac{3}{4}} \end{cases}$

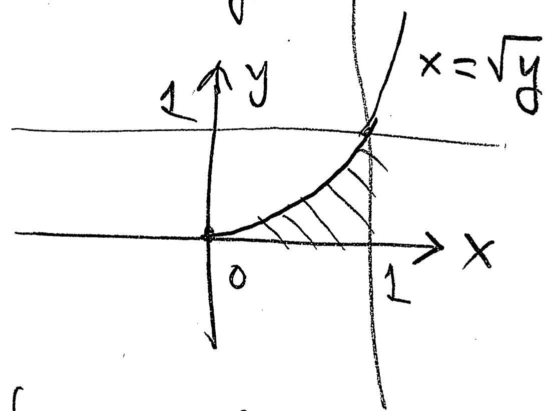
$\therefore$ $I = \int_{-\sqrt{\frac{3}{4}}}^{\sqrt{\frac{3}{4}}} \left(\int_0^{-4y^2+3} (x^3 y)\,dx \right) dy$

$= \int_{-\sqrt{\frac{3}{4}}}^{\sqrt{\frac{3}{4}}} \frac{1}{4}(-4y^2+3)^4 y\,dy = \int_0^0 \frac{1}{4} u^4 \left(-\frac{1}{8}du\right) = 0$ ✓

$u = -4y^2+3$
$du = -8y\,dy$

page 293 #5: $I=\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$

$D: \begin{cases} \sqrt{y} \le x \le 1 \\ 0 \le y \le 1 \end{cases} \Rightarrow$



D is x-simple, ~~but~~

D is also y-simple: $D: \begin{cases} 0 \le y \le x^2 \\ 0 \le x \le 1 \end{cases}$

$$\therefore\ I=\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy=\int_0^1\left(\int_0^{x^2} e^{x^3}\,dy\right)dx=\int_0^1 x^2e^{x^3}\,dx=\frac{1}{3}(e-1)\ ✓$$

page 302 #1

(a) $\int_0^2\int_0^3\int_{-\sqrt{9-x}}^{\sqrt{9-x}} dy\,dz\,dx = \iiint_W dy\,dz\,dx$, where

$W: \begin{cases} -\sqrt{9-x} \le y \le \sqrt{9-x} \\ 0 \le z \le 3 \\ 0 \le x \le 2 \end{cases}$

W bounded on left & right by these surfaces

This is the projection of W onto xz-plane

$\therefore\ \boxed{(ii)}$ ✓

(cont'd)

(b) $\int_0^2 \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx\,dz = \iiint_W dy\,dx\,dz$, where

$$W: \begin{cases} -\sqrt{9-x^2} \le y \le \sqrt{9-x^2} \\ \boxed{\begin{matrix} 0 \le x \le 3 \\ 0 \le z \le 2 \end{matrix}} \end{cases}$$

$\leftarrow$ W bounded on left & right by these surfaces

$\leftarrow$ Projection of W onto xz plane

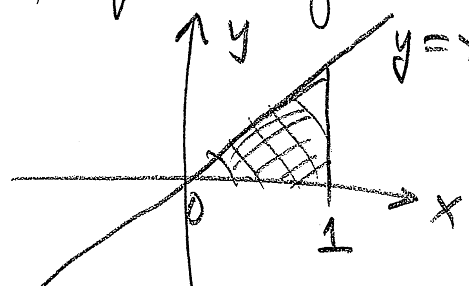
$\therefore$ $\boxed{\text{(i)}}$ ✓

c) $\int_0^1 \int_0^x \int_0^y dz\,dy\,dx = \iiint_W dz\,dy\,dx$, where

$$W: \begin{cases} 0 \le z \le y \\ \boxed{\begin{matrix} 0 \le y \le x \\ 0 \le x \le 1 \end{matrix}} \end{cases}$$

$\leftarrow$ W bounded above and below by these surfaces.

$\leftarrow$ Projection of W onto xy-plane:



$\therefore$ $\boxed{\text{(iii)}}$

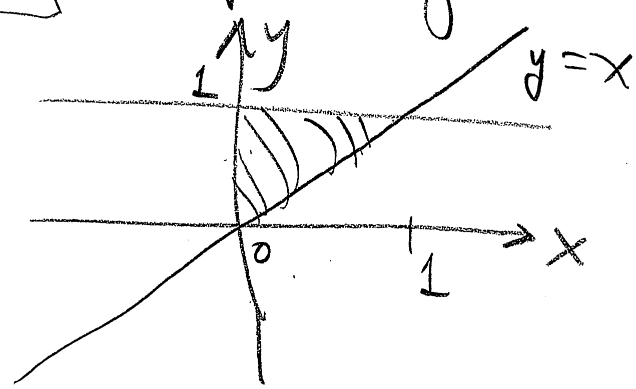
(cont'd)

(d) $\int_0^1 \int_0^y \int_0^x dz\, dx\, dy = \iiint_W dz\, dx\, dy$, where

$$W: \begin{cases} 0 \le z \le x \\ \boxed{\begin{array}{l} 0 \le x \le y \\ 0 \le y \le 1 \end{array}} \end{cases}$$

W bounded above & below by these surfaces

Projection of W onto xy-plane:



$\therefore$ $\boxed{(iv)}$