SOLUTIONS

Homework Set # 11 (PRACTICE)

1. Compute the line integral $\int_C \tilde{\mathbf{F}} \bullet d\tilde{\mathbf{s}}$, where $\tilde{\mathbf{F}} = (2xe^{2y} - \cos y + 2)\,\mathbf{i} + (2x^2e^{2y} + x\sin y)\,\mathbf{j}$ and C is any smooth curve starting at $(1,0)$ and ending at $(2,0)$. (Hint: Is $\tilde{\mathbf{F}}$ a gradient field ?)

2. Find the mass of a wire in the shape of a helix $\mathbf{c}(t) = (\cos 2t,\, \sin 2t,\, t)$ for $0 \le t \le \frac{\pi}{2}$, if the density is $f(x,y,z) = 16y$.

3. Compute the area of the surface S: $\Phi(u,v) = (u\cos v,\, u\sin v,\, v)$, where $D: 0 \le u \le \sqrt{8}, 0 \le v \le u$.

4. Compute the surface integrals $\iint_S x\, dS$ and $\iint_S \tilde{\mathbf{F}} \bullet d\tilde{\mathbf{S}}$, where $\tilde{\mathbf{F}}(x,y,z) = (z,\, 4x,\, 2y+1)$ and S is that part of the plane $\frac{x}{2} + y + z = 1$ in the 1^{st} octant with upward normal.

5. Compute the flux of $\tilde{\mathbf{F}}$ across S, where $\tilde{\mathbf{F}}(x,y,z) = y\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}$ and S is that part of the paraboloid $z = 9 - x^2 - y^2$ which lies above the plane $z = 5$ and $\tilde{\mathbf{N}}$ is the upward unit normal. What is the value of $\iint_S \left(\tilde{\mathbf{F}} \bullet \tilde{\mathbf{N}}\right) dS$?

6. Evaluate the line integral $\int_C x^2y\,dx + y^2\,dy$, along the simple closed curve C that is the positively oriented boundary of the region between $y = 4 - x^2$ and $y = 0$.

7. Compute the line integral $\int_C (-5xy)\;dy + (x^3 + \cos^2 x - 4y)\,dx$, where C is the positively oriented boundary of the rectangle $R = [0,\,2] \times [0,\,3]$.

8. Using **Green's Theorem** find the value of the line integral $\int_C y\,dx + (x^2 + y^2)\,dy$, where C is the circle $(x-3)^2 + y^2 = 9$ traversed in a positive direction:

   Hint: Use fact that the centroid of the region bounded by C is $(\overline{x},\, \overline{y}) = (3,\, 0)$

9. Let S be the surface $z = x^2 + y^2$ below $z = 4$ and downward normal $\mathbf{n}$ and $\tilde{\mathbf{F}}(x,y,z) = (-yz,\, xz,\, y)$. Compute $\iint_S \left(\nabla \times \tilde{\mathbf{F}}\right) \cdot d\tilde{\mathbf{S}}$, using **Stokes' Theorem**: $\iint_S \left(\nabla \times \tilde{\mathbf{F}}\right) \cdot d\tilde{\mathbf{S}} = \int_C \tilde{\mathbf{F}} \bullet d\tilde{\mathbf{s}}$.

10. Let $\tilde{\mathbf{F}} = x\,\mathbf{i} + y\,\mathbf{j} - 3z\,\mathbf{k}$. Compute $\iint_S \tilde{\mathbf{F}} \bullet d\tilde{\mathbf{S}}$, where S is the closed surface consisting of that part of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$, including the top, with outward normal. Compute $\iint_S \tilde{\mathbf{F}} \bullet d\tilde{\mathbf{S}}$ directly by using the **Divergence Theorem.**

11. (§8.3) **Page 459**: # 1.

Solutions

1 $\vec{F}(x,y) = (2xe^{2y}\cos y + 2,\ 2x^2e^{2y} + x\sin y) \stackrel{?}{=} \nabla f(x,y)$

Yes (using MA 366 / Student method): $f(x,y) = x^2e^{2y} - x\cos y + 2x + C$

$\therefore \int_C \vec{F}\cdot d\vec{s} = \int_C \nabla f\cdot d\vec{s} = f(2,0) - f(1,0) = \boxed{4}$

Fundamental Thm of Calculus for Gradient Fields

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2 $\vec{c}(t) = (\underset{x}{\cos 2t}, \underset{y}{\sin 2t}, \underset{z}{t}),\quad 0 \le t \le \frac{\pi}{2} \Rightarrow ds = \|\vec{c}\,'(t)\|dt = \sqrt{5}\,dt$ ✓

$f(x,y,z) = 16y \Rightarrow f(\vec{c}(t)) = 16\sin 2t$

$\therefore M = \int_C 16y\,ds = \int_0^{\frac{\pi}{2}} (16\sin 2t)\sqrt{5}\,dt = \boxed{16\sqrt{5}}$

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3 $S: \Phi(u,v) = (u\cos v, u\sin v, v)$, $D: 0 \le v \le u$, $0 \le u \le \sqrt{8}$

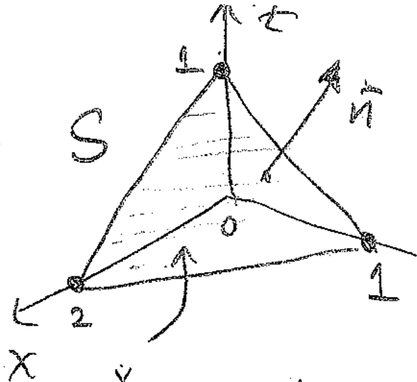
$\Phi_u \times \Phi_v = (\cos v, \sin v, 0) \times (-u\sin v, u\cos v, 1) = (-u\sin v, u\cos v, 1)$

$\therefore dS = \|\Phi_u \times \Phi_v\|\,du\,dv = \sqrt{1+u^2}\,du\,dv$ ✓

$\therefore A(S) = \iint_S dS = \iint_D \sqrt{1+u^2}\,du\,dv = \int_0^{\sqrt{8}}\left(\int_0^u \sqrt{1+u^2}\,dv\right)du$

$= \int_0^{\sqrt{8}} u\sqrt{1+u^2}\,du = \boxed{\frac{26}{3}}$

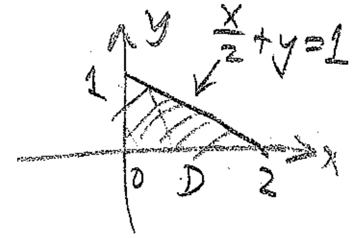
$\boxed{4}$ ②



$\frac{x}{2}+y+z=1$

$$S : \begin{cases} x = x \\ y = y \\ z = 1-y-\frac{x}{2} \end{cases}$$

where $(x,y) \in D$



$\Phi(x,y) = \left(x, y, 1-y-\frac{x}{2}\right) \Rightarrow \Phi_x \times \Phi_y = \left(\frac{1}{2}, 1, 1\right)$ ← correct normal direction!

$\therefore dS = \|\Phi_x \times \Phi_y\| dxdy = \frac{3}{2}\, dxdy$ ✓

$d\vec{S} = (\Phi_x \times \Phi_y)\, dxdy = \left(\frac{1}{2}, 1, 1\right) dxdy$ ✓

Thus, $\iint_S x\, dS = \iint_D x \left(\frac{3}{2} dxdy\right) = \int_0^2 \left(\int_0^{1-\frac{x}{2}} x \; dy\right) dx = \boxed{1}$ ✓

Next, since $\vec{F}(x,y,z) = (z, 4x, 2y+1)$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(1-y-\frac{x}{2}, 4x, 2y+1\right) \cdot \left(\frac{1}{2}, 1, 1\right) dxdy$$

$$= \int_0^2 \left(\int_0^{1-\frac{x}{2}} \left(\frac{3}{2} + \frac{15x}{4} + \frac{3y}{2}\right) dy\right) dx = \boxed{\frac{9}{2}}$$ ✓

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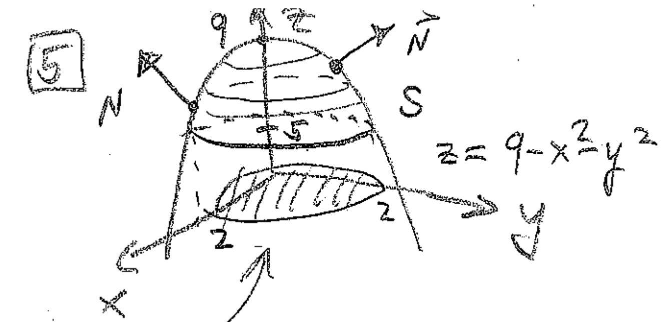
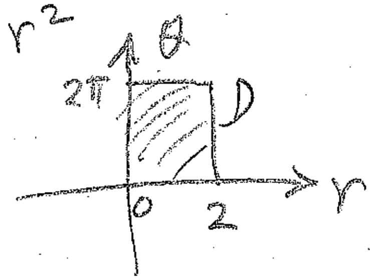


$\therefore$ let $S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 9 - r^2 \end{cases}$

where $(r,\theta) \in D$:



projection of S onto x-y plane is disk of radius 2 (intersection of $z = 9 - x^2 - y^2$ with $z = 5$)

$\Phi_r \times \Phi_\theta = (2r^2\cos\theta, 2r^2\sin\theta, r)$ ✓ ← correct normal direction!

$\vec{F}(x,y,z) = (y, -x, z)$

$\therefore$ Flux of $\vec{F}$ across S is $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(r,\theta))\cdot(\Phi_r \times \Phi_\theta)\,dr\,d\theta$

$= \iint_D (r\sin\theta, -r\sin\theta, 9 - r^2)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 (9r - r^3)\,dr\,d\theta = \boxed{28\pi}$

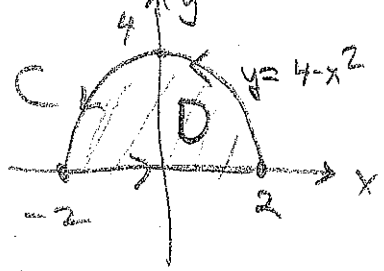
Note that since $\vec{N} = \dfrac{\Phi_r \times \Phi_\theta}{\|\Phi_r \times \Phi_\theta\|}$

$\Rightarrow \iint_S (\vec{F}\cdot\vec{N})\,dS = \iint_D \vec{F}(\Phi)\cdot\left\{\dfrac{\Phi_r \times \Phi_\theta}{\cancel{\|\Phi_r \times \Phi_\theta\|}}\right\}\cancel{\|\Phi_r \times \Phi_\theta\|}\,dr\,d\theta$

$= \iint_S \vec{F}\cdot d\vec{S} = \boxed{28\pi}$, same as above

$\boxed{6}$ $I = \int_C \underbrace{x^2 y}_{P}\, dx + \underbrace{y^3}_{Q}\, dy \underset{\uparrow}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D \left(0 - (x^2)\right) dx\,dy$ (4)

Green's Thm

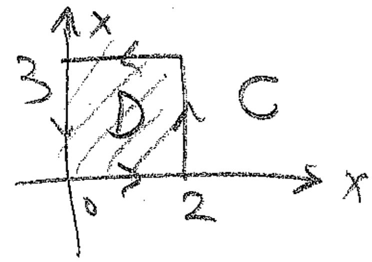


$= \iint_D -x^2\, dx\,dy = \int_{-2}^{2} \int_0^{4-x^2} -x^2\, dy\, dx$

$= \boxed{\dfrac{-128}{15}}$

$\boxed{7}$ $I = \int_C \underbrace{(-5xy)}_{Q}\, dy + \underbrace{(x^3 + \cos^2 x - 4y)}_{P}\, dx \underset{\uparrow}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Green's Thm

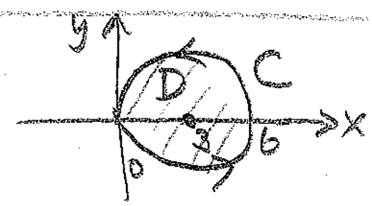


$= \iint_D \left(-5y - (-4)\right) dx\,dy$

$= \int_0^2 \int_0^3 (-5y + 4)\, dy\, dx = \boxed{-21}$

$\boxed{8}$ $I = \int_C y\, dx + (x^2 + y^2)\, dy \underset{\uparrow}{=} \iint_D (2x - 1)\, dx\,dy$

Green's Thm



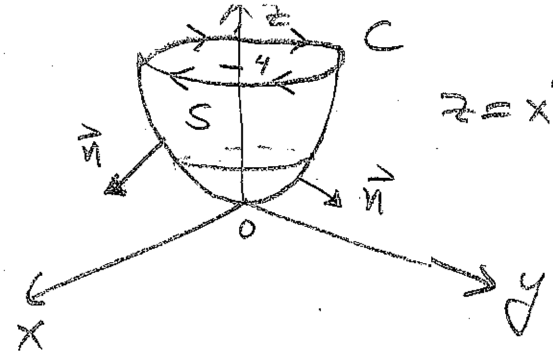
$= 2\iint_D x\, dx\,dy - \iint_D dx\,dy$, Now recall centroid of region D is $\bar{x} = \dfrac{\iint_D x\, dx\,dy}{\iint_D dx\,dy} = \dfrac{\iint_D x\, dx\,dy}{\text{Area } D}$

$\therefore I = 2(\bar{x})(\text{area } D) - (\text{area } D)$

$= 2(3)\left(\pi (3)^2\right) - \pi (3)^2$

$= \boxed{45\pi}$

[9]



$z = x^2 + y^2$

$\vec{F}(x,y,z) = (-yz,\ xz,\ y)$

Stokes's Thm $\Rightarrow \displaystyle\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \underbrace{\int_C \vec{F} \cdot d\vec{s}}_{\text{Much easier to calculate!}}$

the bounday curve C is a circle of radius 2 on the plane $z = 4$, but because $\vec{n}$ is downward $\Rightarrow$ C is traversed in a negative sense (as shown).

Thus a ~~parameterization~~ for $-C$ is as follows

$-C$: $\vec{c}(t) = (2\cos t,\ 2\sin t,\ 4)$, $0 \le t \le 2\pi$

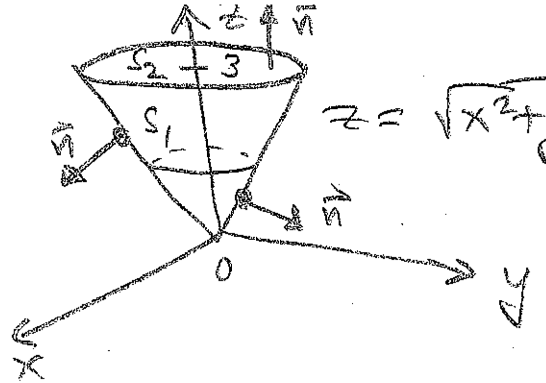
$$\therefore \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{s} = -\left\{ \int_{-C} \vec{F} \cdot d\vec{s} \right\}$$

$$= -\left\{ \int_0^{2\pi} (-8\sin t,\ 8\cos t,\ 2\sin t) \cdot (-2\sin t,\ 2\cos t,\ 0)\, dt \right.$$

$$= -\left\{ \int_0^{2\pi} 16\, dt \right\} = \boxed{-32\pi}$$

Note: You can parameterize the surface S, compute $(\nabla \times \vec{F})$ and then compute surface integral $\displaystyle\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ to get same answer.

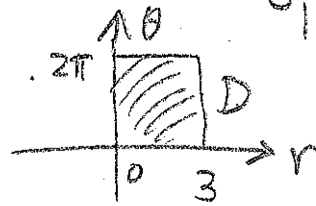
$\boxed{10}$



$z = \sqrt{x^2+y^2}$ $\quad \vec{F}(x,y,z) = (x, y, -3z)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

Now $S_1 : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases}$ where D:



$\Phi(r,\theta) = (\underbrace{r\cos\theta}_{=x}, \underbrace{r\sin\theta}_{=y}, \underbrace{r}_{=z}) \Rightarrow \Phi_r \times \Phi_\theta = (-r\cos\theta, -r\sin\theta, r)$

wrong normal direction! (z-component should be negative)

No problem, just use

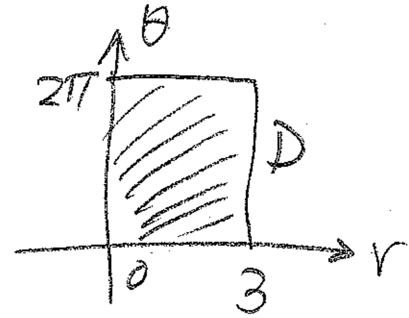
$\Phi_\theta \times \Phi_r = (r\cos\theta, r\sin\theta, -r)$

correct direction now

$$\therefore \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_D (r\cos\theta, r\sin\theta, -3r)\bullet(r\cos\theta, r\sin\theta, -r)\, dr\, d\theta$$

$$= \iint_D 4r^2\, dr\, d\theta = \boxed{72\pi}$$

Next, $S_2 : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 3 \end{cases}$ where $(r,\theta)\in D$:



$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 3)$ and so $\Phi_r \times \Phi_\theta = (0, 0, r)$

correct normal direction

(cont'd)

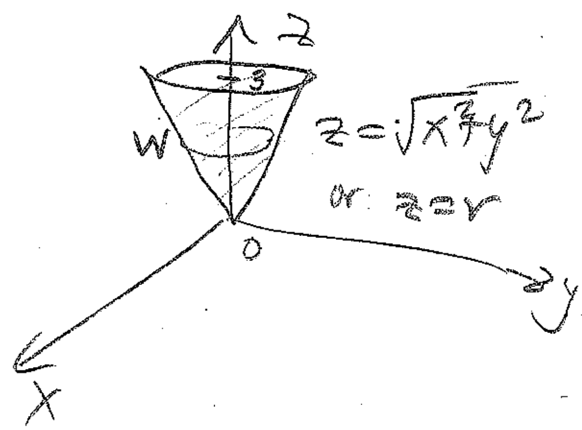
$F(x,y,z) = (x, y, -3z)$

Thus $\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D (r\cos\theta, r\sin\theta, -9) \cdot (0,0,r)\, dr\, d\theta$

$= \iint_D -9r\, dr\, d\theta = \boxed{-81\pi}$

Thus $\iint_S \vec{F} \cdot d\vec{S} = (72\pi) + (-81\pi) = \boxed{-9\pi}$ ✓

Now, using the Divergence Thm,
(since S is a closed surface)



$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W (\operatorname{div} \vec{F})\, dx\, dy\, dz$; $\operatorname{div} \vec{F} = 1 + 1 - 3 = -1$

$= \iiint_W (-1)\, dx\, dy\, dz$ $\left(= (-1) \text{ Volume of Cone}\right)$

$= \int_0^{2\pi} \int_0^3 \int_r^3 r\, dz\, dr\, d\theta = \boxed{-9\pi}$, Same as above

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Recall, $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is conservative

if $\boxed{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$

$\therefore$ (a) $\vec{F}(x,y) = (x, y)$, P, Q. Check: $\dfrac{\partial Q}{\partial x} \stackrel{?}{=} \dfrac{\partial P}{\partial y}$

$0 = 0$ ✓

$\boxed{\text{YES}}$ and $f(x,y) = \frac{1}{2}x^2 + y^2 + C$ (MA 366/Student method)

(b) $\vec{F}(x,y) = (xy, xy)$, P, Q. Check: $\dfrac{\partial Q}{\partial x} \stackrel{?}{=} \dfrac{\partial P}{\partial y}$

$\boxed{\text{NO}}$ $y \neq x$

(c) $\vec{F}(x,y) = (x^2+y^2, 2xy)$. Check $Q_x \stackrel{?}{=} P_y$, P, Q

$2y = 2y$ ✓

$\boxed{\text{YES}}$ MA 366/Student method gives $f(x,y) = \frac{1}{3}x^3 + xy^2 + C$

Note: If $\vec{F}$ is a vector field on $\mathbb{R}^3$, then $\vec{F}$ is conservative if $\nabla \times \vec{F} = \vec{0}$ (i.e. curl $\vec{F} = \vec{0}$)