Basic Differentiation Rules for Paths

Let $\mathbf{b}, \mathbf{c} : [a, b] \longrightarrow \mathbb{R}^n$ be differentiable paths in $\mathbb{R}^n$ and let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued differentiable function, then

- $\boxed{\text{i}}$ $\big[\mathbf{b}(t) + k\,\mathbf{c}(t)\big]' = \mathbf{b}'(t) + k\,\mathbf{c}'(t), \ \forall k \in \mathbb{R}$.
- $\boxed{\text{ii}}$ $\big[\phi(t)\,\mathbf{c}(t)\big]' = \phi(t)\,\mathbf{c}'(t) + \phi'(t)\,\mathbf{c}(t)$.
- $\boxed{\text{iii}}$ $\big[\mathbf{b}(t) \bullet \mathbf{c}(t)\big]' = \mathbf{b}(t) \bullet \mathbf{c}'(t) + \mathbf{b}'(t) \bullet \mathbf{c}(t)$.
- $\boxed{\text{iv}}$ If $n = 3$, then $\big[\mathbf{b}(t) \times \mathbf{c}(t)\big]' = \mathbf{b}(t) \times \mathbf{c}'(t) + \mathbf{b}'(t) \times \mathbf{c}(t)$.
- $\boxed{\text{v}}$ $\big[\mathbf{c}(\phi(t))\big]' = \mathbf{c}'(\phi(t))\,\phi'(t)$.

Some Basic Identities for Curl and Divergence

Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be real-valued functions and $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be vector fields in $\mathbb{R}^n$.

Then, under appropriate differentiability conditions :

- $\boxed{1}$ $\nabla(f + k\,g) = \nabla f + k\,\nabla g, \ \forall k \in \mathbb{R}$.
- $\boxed{2}$ $\nabla(f\,g) = f\,\nabla g + g\,\nabla f$.
- $\boxed{3}$ $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\,\nabla f - f\,\nabla g}{g^2}$, whenever $g(\mathbf{x}) \neq 0$.
- $\boxed{4}$ $\text{div}\,\big(\mathbf{F} + \mathbf{G}\big) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$.
- $\boxed{5}$ $\text{div}\,\big(f\,\mathbf{F}\big) = f\,\text{div}\,\mathbf{F} + \mathbf{F} \bullet \nabla f$.
- $\boxed{6}$ $\text{div}\,\big(\mathbf{F} \times \mathbf{G}\big) = \mathbf{G} \bullet \text{curl}\,\mathbf{F} - \mathbf{F} \bullet \text{curl}\,\mathbf{G}$.
- $\boxed{7}$ $\text{div}\,(\text{curl}\,\mathbf{F}) = 0$.
- $\boxed{8}$ $\text{curl}\,(\nabla f) = \vec{\mathbf{O}}$.
- $\boxed{9}$ $\text{curl}\,(\mathbf{F} + \mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$.
- $\boxed{10}$ $\text{curl}\,\big(f\,\mathbf{F}\big) = f\,\text{curl}\,\mathbf{F} + \nabla f \times \mathbf{F}$.
- $\boxed{11}$ $\text{div}\,\big(\nabla f \times \nabla g\big) = 0$.

⚠ Any identity above that involves **curl** holds only when $n = 3$.

<u>**Remarks**</u> :

(a) In $\mathbb{R}^3$, the *Del Operator* is $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$

(b) Divergence & Curl in Del Notation: $\boxed{\text{div}\,\mathbf{F} = \nabla \bullet \mathbf{F}}$ and $\boxed{\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}}$

(c) $\nabla^2 f = \nabla \bullet (\nabla f) = \dfrac{\partial^2 f}{\partial x_1^2} + \dfrac{\partial^2 f}{\partial x_2^2} + \cdots + \dfrac{\partial^2 f}{\partial x_n^2}$. This is the *Laplacian* of f.

(d) A path $\mathbf{c}(t)$ is a <u>flow line</u> for a vector field $\mathbf{F}$ if $\boxed{\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))}$.

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