Basic Differentiation Rules

Let $\mathbf{f} : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbf{g} : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at $\mathbf{x}_0 \in U$, then

i *Constant Multiple Rule* : $\mathbf{D}(c\mathbf{f})(\mathbf{x}_0) = c\,\mathbf{Df}(\mathbf{x}_0)$, for all $c \in \mathbb{R}$.

ii *Sum Rule* : $\mathbf{D}(\mathbf{f}+\mathbf{g})(\mathbf{x}_0) = \mathbf{Df}(\mathbf{x}_0) + \mathbf{Dg}(\mathbf{x}_0)$.

iii *Product Rule* : If $m = 1$, then $\mathbf{D}(fg)(\mathbf{x}_0) = f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0) + g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0)$.

iv *Quotient Rule* : If $m = 1$, then $\mathbf{D}\left(\frac{f}{g}\right)(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) - f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0)}{\{g(\mathbf{x}_0)\}^2}$, provided $g \neq 0$ on U.

Note that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $\mathbf{D}f(\mathbf{x}) = \nabla f(\mathbf{x})$.

<u>CHAIN RULE</u> - Let $\mathbf{g} : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbf{f} : V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^p$, with U and V open sets and so that $f \circ g$ is defined. If $\mathbf{g}$ is differentiable at $\mathbf{x}_0 \in U$ and $\mathbf{f}$ is differentiable at $\mathbf{g}(\mathbf{x}_0)$, then $\mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{x}_0$ and

$$\boxed{\mathbf{D}(\mathbf{f} \circ \mathbf{g})(\mathbf{x}_0) = \big[\mathbf{Df}(\mathbf{g}(\mathbf{x}_0))\big]\,\big[\mathbf{Dg}(\mathbf{x}_0)\big]}$$

<u>Note</u>: If $\mathbf{f}$: $V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^p$, so that $\mathbf{f}(\mathbf{x}) = \Big(f_1(\mathbf{x}),\ f_2(\mathbf{x}), \cdots, f_p(\mathbf{x})\Big)$ and $\mathbf{x} = (x_1, x_2, \cdots, x_m)$, then $\mathbf{D}f$ is the $p \times m$ matrix of partial derivatives of $\mathbf{f}$:

$$\mathbf{D}f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1} & \frac{\partial f_p}{\partial x_2} & \cdots & \frac{\partial f_p}{\partial x_m} \end{bmatrix}_{p\times m}$$

<u>Special Case of Chain Rule</u> - Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued differentiable function of n variables and $\mathbf{c} : \mathbb{R} \longrightarrow \mathbb{R}^n$ a differentiable path in $\mathbb{R}^n$ so that $f \circ \mathbf{c}$ is defined, then

$$\boxed{\mathbf{D}(f \circ \mathbf{c})(t) = \mathbf{D}(f(\mathbf{c}(t)))\,\mathbf{Dc}(t) = \nabla f(\mathbf{c}(t)) \bullet \mathbf{c}'(t)}$$