Limit Theorems

Basic Properties of Limits - Let $\mathbf{f} : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbf{g} : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $\mathbf{x}_0 \in A$ or a boundary point of A. If $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{b}_1$ and $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x}) = \mathbf{b}_2$, then

1. $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} c\,\mathbf{f}(\mathbf{x}) = c \lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = c\,\mathbf{b}_1$, for all $c \in \mathbb{R}$.

2. $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \{\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\} = \left\{\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})\right\} + \left\{\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x})\right\} = \mathbf{b}_1 + \mathbf{b}_2$.

3. If $m = 1$, then $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \{f(\mathbf{x})\,g(\mathbf{x})\} = \left\{\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\right\}\left\{\lim\limits_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})\right\} = b_1 b_2$.

4. If $m = 1$, then $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \left\{\dfrac{f(\mathbf{x})}{g(\mathbf{x})}\right\} = \dfrac{\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})} = \dfrac{b_1}{b_2}$, provided $b_2 \neq 0$.

5. If $\mathbf{f}(\mathbf{x}) = \Big(f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_m(\mathbf{x})\Big)$ and $\mathbf{b} = \big(b_1, b_2, \cdots, b_m\big)$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{\mathbf{x}\to\mathbf{x}_0} f_k(\mathbf{x}) = b_k, \quad \text{for each} \quad k = 1, 2, \cdots, m.$$

Limit Composition Theorem - Let $\mathbf{g} : B \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$ and $\mathbf{f} : A \subset \mathbb{R}^k \longrightarrow \mathbb{R}^m$ so that $f \circ g$ is defined. If $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x}) = \mathbf{a}$ and $\mathbf{f}$ is continuous at $\mathbf{a}$, then $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}\left(\mathbf{g}(\mathbf{x})\right) = \mathbf{f}\left(\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x})\right) = \mathbf{f}(\mathbf{a})$.