MA 366
REVIEW - TEST # 2

(0) Results from Calculus: differentiation formulas, implicit differentiation, Chain Rule; integration formulas, integration by parts, partial fractions; other integration techniques; Solutions of First Order Linear or Separable Equations.

(1) Second order equations; rewriting 2nd order equations as an equivalent system of 1st order equations.

(2) Theorem 1 Existence & Uniqueness for 2nd Order Linear Equations: If \( p(t), q(t) \) and \( g(t) \) are continuous for \( \alpha < t < \beta \), containing \( t_0 \), then the initial value problem \( y'' + p(t)y' + q(t)y = g(t) \), \( y(t_0) = y_0 \), \( y'(t_0) = y_1 \) has a unique solution defined for \( \alpha < t < \beta \).

(The largest such interval is the interval of existence for the solution.)

(3) Theorem 2 Superposition Principle: If \( y_1(t) \) and \( y_2(t) \) are solutions to the linear homogeneous equation \( a_0(t)y'' + a_1(t)y' + a_2(t)y = 0 \), then any linear combination of \( y_1(t) \) and \( y_2(t) \) is also a solution i.e., \( y = C_1y_1(t) + C_2y_2(t) \) is also a solution for any constants \( C_1 \) and \( C_2 \).

(4) Linear independence/dependence of function \( u(t) \) and \( v(t) \); Wronskian \( W(t) \) of \( y_1 \) and \( y_2 \); Wronskians and solutions to equations:

\[
\text{Theorem 3} \quad \text{If } y_1(t) \text{ and } y_2(t) \text{ are solutions to } a_0(t)y'' + a_1(t)y' + a_2(t)y = 0, \text{ then either } W(t) \equiv 0 \text{ or } W(t) \neq 0.
\]

(5) Fundamental Set of Solutions (FSS) of a 2nd order equation; general solution of a 2nd order equation:

\[
\text{Theorem 4} \quad \text{General Solution of Homogeneous Equations}
\]
\[
\text{If } y_1(t) \text{ and } y_2(t) \text{ are linearly independent solutions to the 2nd order linear homogeneous equation } a_0(t)y'' + a_1(t)y' + a_2(t)y = 0, \text{ then the general solution is given by } y = C_1y_1(t) + C_2y_2(t).
\]

(6) Solutions of linear homogeneous equations with constant coefficients; characteristic equation, characteristic polynomial, characteristic roots.

(7) Reduction of Order: If \( y_1(t) \) is a solution to the linear homogeneous equation \( a_0(t)y'' + a_1(t)y' + a_2(t)y = 0 \), then a second solution has the form \( y = y_1(t)v(t) \).

(This reduces the equation to a 1st order equation in the variable \( v' \).)

(8) Elementary Spring-Mass Problems: \( my'' + \mu y' + ky = F(t) \); Simple harmonic motion \((\mu = 0 \text{ and } F(t) = 0)\), hence \( y(t) = A\cos(\omega_0 t - \phi) \), amplitude, natural frequency, phase angle; Damped harmonic motion \((\mu \neq 0 \text{ and } F(t) = 0)\), overdamped, underdamped, critically damped.
Theorem 5  General Solution of Nonhomogeneous Equations
If $y_p(t)$ is a particular solution to the nonhomogeneous equation

$$a_0(t)y'' + a_1(t)y' + a_2(t)y = g(t) \quad (*)$$

and $y_h(t)$ is the general solution to the corresponding homogeneous equation

$$a_0(t)y'' + a_1(t)y' + a_2(t)y = 0,$$

then the general solution to the nonhomogeneous equation $(*)$ is $y = y_h(t) + y_p(t)$.

Particular solutions $y_p(t)$: You can always use the method of Variation of Parameters to find a particular solution $y_p(t)$ to the linear nonhomogeneous equation $a_0(t)y'' + a_1(t)y' + a_2(t)y = g(t)$. However, if the coefficients are constants rather than functions AND $g(t)$ has a very special form (see table below), it is usually easier to use Undetermined Coefficients:

(a) **Undetermined Coefficients** - Use only IF the equation has the form $ay'' + by' + cy = g(t)$ AND $g(t)$ is as below:

<table>
<thead>
<tr>
<th>$g(t)$</th>
<th>Form of $y_p(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$</td>
<td>$t^s \left( A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0 \right)$</td>
</tr>
<tr>
<td>$e^{\alpha t} P_n(t)$</td>
<td>$t^s e^{\alpha t} \left( A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0 \right)$</td>
</tr>
<tr>
<td>$e^{\alpha t} \left{ \sin \beta t \quad \text{or} \quad \cos \beta t \right}$</td>
<td>$t^s e^{\alpha t} \left[ F_n(t) \cos \beta t + G_n(t) \sin \beta t \right]$</td>
</tr>
</tbody>
</table>

where $s = \text{the smallest nonnegative integer such that no term of } y_p \text{ is a solution to the corresponding homogeneous equation (} s = 0, 1 \text{ or } 2 \text{)}$ in other words, no term of $y_p(t)$ is a term of $y_h(t)$. ($F_n(t), G_n(t)$ are polynomials of degree $n$.)

(b) **Variation of Parameters** - Given $y'' + p(t)y' + q(t)y = g(t)$ and given two linearly independent solutions $y_1(t)$ and $y_2(t)$ to the corresponding homogeneous equation, then a particular solution $y_p(t)$ has the form

$$y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$$

where

$$v'_1 = \begin{vmatrix} 0 & y_2 \\ g(t) & y'_2 \end{vmatrix} \quad \text{and} \quad v'_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & g(t) \end{vmatrix},$$

and $W(t) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$ is the Wronskian of $y_1$ and $y_2$. 


**Useful Remark**: If $y_{p_1}(t)$ is a particular solution of
\[ a_0(t)y'' + a_1(t)y' + a_2(t)y = g_1(t) \]
and $y_{p_2}(t)$ is a particular solution of
\[ a_0(t)y'' + a_1(t)y' + a_2(t)y = g_2(t) , \]
then
\[ y_p(t) = y_{p_1}(t) + y_{p_2}(t) \]
is a particular solution of
\[ a_0(t)y'' + a_1(t)y' + a_2(t)y = g_1(t) + g_2(t) . \]

(11) **Euler(Tangent Line) Method.** You should be able to compute by hand the first few approximations to the true solution $\phi(t)$ of the IVP
\[
\begin{aligned}
\frac{dy}{dt} &= f(t, y) \\
y(t_0) &= y_0
\end{aligned}
\]
Thus $y_n \approx \phi(t_n)$, where
\[ y_n = y_{n-1} + h f(t_{n-1}, y_{n-1}) \]
and $t_k = t_0 + kh$, and $h$ is the step-size.

**Remark.** You can tell if the Euler approximation is *smaller* or *larger* than the actual solution $\phi(t)$ near $t_0$ by looking at the sign of $\frac{d^2y}{dt^2}$ at $t_0$:
\[ \frac{d^2y}{dt^2} > 0 \text{ at } t_0 \implies \text{EULER approximation } < \phi(t) \text{ near } t_0 \]
\[ \frac{d^2y}{dt^2} < 0 \text{ at } t_0 \implies \text{EULER approximation } > \phi(t) \text{ near } t_0 \]

(12) Systems of differential equations; direction field of a planar system $\begin{cases} x' = f(t,x,y) \\
y' = g(t,x,y) \end{cases}$; autonomous systems $\begin{cases} x' = f(x,y) \\
y' = g(x,y) \end{cases}$; equilibrium points of autonomous systems (i.e., where $x' = y' = 0$).

(13) Linear systems of equations; matrix notation; applications of linear systems to multiple springs and mixing problems with multiple tanks.

(14) Solution to linear homogeneous systems of the form $x' = Ax$, that is
\[
\begin{cases}
x_1' = ax_1 + bx_2 \\
x_2' = cx_1 + dx_2
\end{cases}
\text{ or } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
using the **Elimination Method**.
Finding eigenvalues $\lambda$ of a square matrix $A$: $\det(A - \lambda I) = 0$; computing their corresponding eigenvectors $\vec{v} \neq \vec{0}$: $(A - \lambda I)\vec{v} = \vec{0}$; characteristic equation for $A$; solutions of linear systems of the form $x' = Ax$ using eigenvalues and eigenvectors (real and distinct, complex and repeated); eigenspace of $\lambda$.

**Phase Portraits of Linear Homogeneous Systems**: A plot of the solutions (orbits or trajectories) of the homogeneous system

$$\vec{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x}$$

is called a phase portrait. To sketch the phase portrait, we need to find the corresponding eigenvalues of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then consider 3 cases:

(i) $\lambda_1 < \lambda_2$, real and distinct  

If $\vec{v}^{(1)}$, $\vec{v}^{(2)}$ are eigenvectors corresponding to $\lambda_1$ and $\lambda_2$, respectively \Rightarrow $\vec{x}^{(1)}(t) = e^{\lambda_1 t} \vec{v}^{(1)}$ and $\vec{x}^{(2)}(t) = e^{\lambda_2 t} \vec{v}^{(2)}$ are solutions and hence general solution is $\vec{x}(t) = C_1 \vec{x}^{(1)}(t) + C_2 \vec{x}^{(2)}(t)$. 

![Phase Portrait Diagrams](image)
(ii) \( \lambda_1 = \alpha + i \beta \). If \( \vec{w} = \vec{a} + i \vec{b} \) is a complex eigenvector corresponding to \( \lambda_1 \), then \( \vec{x}^{(1)}(t) = \Re e \{ e^{\lambda_1 t} \vec{w} \} = e^{\alpha t} (\vec{a} \cos \beta t - \vec{b} \sin \beta t) \) and the 2nd solution \( \vec{x}^{(2)}(t) = \Im m \{ e^{\lambda_1 t} \vec{w} \} = e^{\alpha t} (\vec{a} \sin \beta t + \vec{b} \cos \beta t) \) are real-valued solutions and hence general solution is

\[
\vec{x}(t) = C_1 \vec{x}^{(1)}(t) + C_2 \vec{x}^{(2)}(t).
\]

If \( \alpha < 0 \):

![Diagram](image1)

(Test a point to decide which)

(iii) \( \lambda_1 = \lambda_2 \) \( \Rightarrow \) Solutions are \( \vec{x}^{(1)}(t) = e^{\lambda_1 t} \vec{v} \) and \( \vec{x}^{(2)}(t) = t e^{\lambda_1 t} \vec{v} + e^{\lambda_1 t} \vec{w} \), where

\[
(A - \lambda_1 I) \vec{v} = 0 \\
(A - \lambda_1 I) \vec{w} = \vec{v}
\]

(\( \vec{v} \) is an eigenvector while \( \vec{w} \) is a “generalized eigenvector”)

The general solution of system is \( \vec{x}(t) = C_1 \vec{x}^{(1)}(t) + C_2 \vec{x}^{(2)}(t) \).

If \( \lambda_1 < 0 \):

![Diagram](image2)

(Test a point to decide which)
Classification of Equilibrium Points: An equilibrium point is a sink if all solutions which begin sufficiently close to it converge to it. It is a source if all solutions sufficiently close to it move away from it. It is a center if all solutions which begin sufficiently close to it “loop around” it, i.e., they return to their initial position after a finite amount of time. Finally, the equilibrium point is a saddle if some solutions converge to it and some move away from it.

Solutions of \( n \)th order linear homogeneous equations with constant coefficients:
\[
a_n y^{(n)} + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 y = 0.
\]

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**Practice Problems**

1. **True/False:**
   
   (a) If \( \{y_1, y_2\} \) is a Fundamental Set of Solutions to \( t^2 y'' - e^{-3t} y' + t^3 y = 0 \), then \( \{2y_1, y_1 - 3y_2\} \) is also a solution.

   (b) The interval of existence for the solution to
   \[
   \begin{align*}
   (t^2 - 2t)y'' + \frac{te^t}{(t + 1)} y &= t^2 \sqrt{9 - t^2} \\
y(1) &= 0 \\
y'(1) &= 2005
   \end{align*}
   \]
   is \( 0 < t < 2 \).

   (c) The Wronskian \( W(t) \) of the functions \( y_1(t) = t^2 \) and \( y_2(t) = t|t| \) satisfies
   \( W(t) = 0 \) for \( -\infty < t < \infty \).

   (d) The functions \( y_1(t) = t^2 \) and \( y_2(t) = t|t| \) are linearly dependent for all values \( -\infty < t < \infty \).

2. The differential equation \( t^2 y'' + ty' - y = 0 \) has one solution \( y_1(t) = t \). Use the method of **Reduction of Order** to find a second (linearly independent) solution of \( t^2 y'' + ty' - y = 0 \).

3. For what nonnegative values of \( \mu \) will the solution of the initial value problem
   \( y'' + \mu y' + 4y = 0, \ y(0) = 4, \ y'(0) = 0 \) oscillate?

4. (a) For what positive values of \( k \) does the solution of the initial value problem
   \( 2y'' + ky = 3\cos(2t), \ y(0) = 0, \ y'(0) = 0 \) become unbounded (Resonance)?

   (b) For what positive values of \( k \) does the solution of the initial value problem
   \( 2y'' + y' + ky = 3\cos(2t), \ y(0) = 0, \ y'(0) = 0 \) become unbounded (Resonance)?

5. Find the steady-state solution of the IVP
   \( y'' + 4y' + 4y = \sin t, \ y(0) = 0, \ y'(0) = 0 \).

6. A 4-kg mass stretches a spring 0.392 m. If the mass is released from 1 m below the equilibrium position with a downward velocity of 10 m/sec, what is the maximum displacement?
In Problems 7, 8 and 9 find the general solution of the homogeneous differential equations in (a), find a particular solution $y_p(t)$ in (b) and use the method of **Undetermined Coefficients** to find the **form** of a particular solution of the nonhomogeneous equations in (c).

7. (a) $y'' - 5y' + 6y = 0$  
    (b) $y'' - 5y' + 6y = 6t^2$  
    (c) $y'' - 5y' + 6y = e^{2t} + \cos(3t)$

8. (a) $y'' - 6y' + 9y = 0$  
    (b) $y'' - 6y' + 9y = te^{3t}$  
    (c) $y'' - 6y' + 9y = e^{-2t} + \cos(3t) - 1$

9. (a) $y'' - 2y' + 10y = 0$  
    (b) $y'' - 2y' + 10y = \cos(3t)$  
    (c) $y'' - 2y' + 10y = t^2e^t \sin(3t)$

10. Find the general solution of the differential equation $y'' - y' = 4t$.

11. Find the general solution to $y'' + y = \tan t$, $0 < t < \frac{\pi}{2}$.

12. The differential equation $x^2y'' - 2xy' + 2y = 0$ has solutions $y_1(x) = x$ and $y_2(x) = x^2$. Use the method of **Variation of Parameters** to find a solution to

\[ x^2y'' - 2xy' + 2y = 2x^2. \]

13. Use Euler’s Method to approximate the solution to the IVP below at $t = 1.6$, using $h = 0.2$. What is the value of the exact solution at $t = 1.6$

\[
\begin{cases} 
  y' = 1 + 2y - 2t^2 \\
  y(1) = 2 
\end{cases}
\]

14. Use the **Elimination Method** to solve the system

\[
\begin{align*}
  x_1' &= x_1 + x_2 \\
  x_2' &= 4x_1 + x_2
\end{align*}
\]

15. Rewrite the 2\textsuperscript{nd} order differential equation $y'' + 2y' + 3ty = \cos t$ with $y(0) = 1, y'(0) = 4$ as a system of 1\textsuperscript{st} order differential equations.

16. Find eigenvalues and corresponding eigenvectors of $A$:

(a) \[
\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}
\]
(b) \[
\begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}
\]
(c) \[
\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]
(d) \[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

17. Find the solution of the IVP \[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.
\]

18. Solve \[
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \bar{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.
\]

19. Find the general solution of the system \[
\bar{x}'(t) = A\bar{x}(t), \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
20. Tank # 1 initially holds 50 gals of brine with a concentration of 1 lb/gal, while Tank # 2 initially holds 25 gals of brine with a concentration of 3 lb/gal. Pure H₂O flows into Tank # 1 at 5 gal/min. The well-stirred solution from Tank # 1 then flows into Tank # 2 at 5 gal/min. The solution in Tank # 2 flows out at 5 gal/min. Set up and solve an IVP that gives \( x_1(t) \) and \( x_2(t) \), the amount of salt in Tanks # 1 and # 2, respectively, at time \( t \).

21. Tank # 1 initially holds 50 gals of brine with concentration of 1 lb/gal and Tank # 2 initially holds 25 gals of brine with concentration 3 lb/gal. The solution in Tank # 1 flows at 5 gal/min into Tank # 2, while the solution in Tank # 2 flows back into Tank # 1 at 5 gal/min. Set up an IVP that gives \( x_1(t) \) and \( x_2(t) \), the amount of salt in Tanks # 1 and # 2, respectively, at time \( t \).

22. Match the phase portraits shown below that best corresponds to each of the given systems of differential equations and classify the equilibrium point \((0,0)\):

(i) \( \mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} \); Solution: \( \mathbf{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \)

(ii) \( \mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{x} \); Solution: \( \mathbf{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \)

(iii) \( \mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} \); Solution: \( \mathbf{x}(t) = C_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^t + C_2 e^t \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \)

(iv) \( \mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x} \); Solution: \( \mathbf{x}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t} \)

A. B.
23. Find all equilibrium points for these systems:

\[(a) \begin{cases} x' = x(2 - 3y) \\ y' = y(-4 + x) \end{cases} \quad (b) \begin{cases} x' = x(1 - y) \\ y' = x^2 - y(x + 6) \end{cases} \quad (c) \begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad (a, b, c, d \in \mathbb{R})\]

24. Find general solutions to these \(n\)th order linear equations with constant coefficients:

\[(a) y''' = 0 \quad (b) y^{(4)} - 16y = 0 \quad (c) y''' - 2y'' - y' + 2y = 0\]

25. If \(\lambda\) is an eigenvalue of the \(n \times n\) matrix \(A\), prove that \(3\lambda\) is an eigenvalue of \(3A\).

26. If \(A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\) and \(\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\), determine \(A^7\mathbf{v}\).

\((Hint: \mathbf{v} \text{ is an eigenvector of } A.)\)

\underline{Answers}

1. (a) True   (b) False   (c) True   (d) False
2. \(y_2(t) = \frac{1}{t}\).
3. \(0 \leq \mu < 4\)
4. (a) \(k = 8\) (resonance)   (b) NO value of \(k\), all solutions are bounded.
5. \(y = \frac{1}{25}(3\sin t - 4\cos t)\)
6. \(y(t) = \cos 5t + 2 \sin 5t = \sqrt{5}\cos(5t - \phi), \ \phi = \cos^{-1}\frac{1}{\sqrt{5}} \approx 1.1\) Thus amplitude = \(\sqrt{5}\).
7. (a) \(y = C_1e^{3t} + C_2e^{2t}\)   (b) \(y_p = t^2 + \frac{5}{3}t + \frac{19}{18}\)   (c) \(y_p = At^2 + B \cos 3t + C \sin 3t\)
8. (a) \( y = C_1 e^{3t} + C_2 t e^{3t} \)  (b) \( y_p = \frac{t^3}{6} e^{3t} \) 
(c) \( y_p = A e^{-2t} + B \cos 3t + C \sin 3t + D \)
9. (a) \( y = C_1 \cos 3t + C_2 t \sin 3t \)  (b) \( y_p = \frac{1}{3} \cos 3t - \frac{6}{37} \sin 3t \) 
(c) \( y_p = e^t (A t^2 + B t^2 + C t) \cos 3t + e^t (D t^3 + E t^2 + F t) \sin 3t \)
10. \( y = C_1 + C_2 e^x - 2 t^2 - 4 t \)
11. \( y = C_1 \cos t + C_2 \sin t - (\cos t) \ln(\sec t + \tan t) \)
12. \( y = 2x^2 \ln x \) (or \( y = 2x^2 \ln x + (C_1 x + C_2 x^2) \))
13. \( y_3 = 3.9856 \approx \phi(1.6) \). Exact solution is \( \phi(t) = t^2 + t \) and hence \( \phi(1.6) = 4.16 \)
14. \( x_1(t) = C_1 e^{3t} + C_2 e^{-t}, \ x_2(t) = 2C_1 e^{3t} - 2C_2 e^{-t} \)
15. Let \( x_1 = y, \ x_2 = y' \), then \( \begin{vmatrix} x_1' = x_2 \\ x_2' = -3tx_1 - 2x_2 + \cos t \end{vmatrix} \), where \( x_1(0) = 1, \ x_2(0) = 4 \)
16. (a) \( \lambda_1 = 3, \ \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \); \( \lambda_2 = -1, \ \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \)
16. (b) \( \lambda_1 = -1, \ \vec{v}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \); \( \lambda_2 = -2, \ \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)
16. (c) \( \lambda_1 = 1 + i, \ \vec{w}^{(1)} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \); \( \lambda_2 = 1 - i, \ \vec{w}^{(2)} = \begin{pmatrix} i \\ 1 \end{pmatrix} \)
(16. many other possibilities !)
16. (d) \( \lambda_1 = \lambda_2 = 1 \) a repeated eigenvalue with only one linearly independent eigenvector \( \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)
17. \( \vec{x}(t) = 2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \)
18. \( \vec{x}(t) = 2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} - e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \)
19. \( \vec{x}(t) = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^t \\ 1 \end{pmatrix} + t e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)
20. \( \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} -\frac{1}{10} \ 0 \\ \frac{7}{10} \ -\frac{1}{5} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}, \ \begin{vmatrix} x_1(0) \\ x_2(0) \end{vmatrix} = \begin{vmatrix} 50 \\ 75 \end{vmatrix} \)
Solution: \( \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = 50 e^{-\frac{1}{5}} \begin{vmatrix} 1 \\ 1 \end{vmatrix} + 25 e^{\frac{1}{5}} \begin{vmatrix} 0 \\ 1 \end{vmatrix} \)
21. \( \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} -\frac{1}{10} \ 0 \\ \frac{7}{10} \ -\frac{1}{5} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}, \ \begin{vmatrix} x_1(0) \\ x_2(0) \end{vmatrix} = \begin{vmatrix} 50 \\ 75 \end{vmatrix} \)
Solution: \( \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \frac{125}{3} e^{\frac{1}{5}} \begin{vmatrix} 2 \\ 1 \end{vmatrix} - \frac{100}{3} e^{-\frac{1}{5}} \begin{vmatrix} 1 \\ -1 \end{vmatrix} \)
22. (i) C (0, 0) saddle (ii) A (0, 0) source (iii) B (0, 0) source (iv) D (0, 0) sink
23. (a) (0, 0), (4, 2) (b) (0, 0), (3, 1), (-2, 1) (c) (0, 0)
24. (a) \( y = C_1 + C_2 t + C_3 t^2 \)  (b) \( y = C_1 e^{2t} + C_2 e^{-2t} + C_3 \cos 2t + C_4 \sin 2t \)
(c) \( y = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} \)
25. \( \det(3A - 3\lambda I) = 3^n \det(A - \lambda I) = 0 \), since \( \lambda \) is an eigenvalue of \( A \).
26. Since \( \vec{v} \) is an eigenvector of \( A \) corresponding to \( \lambda = 3 \), \( A^7 \vec{v} = 3^7 \vec{v} = 3^7 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).