Variation of Parameters

Variation of Parameters (2^{nd} order linear nonhomogeneous equation) :

If $y_1(t)$ and $y_2(t)$ form a FSS of the 2^{nd} order linear homogeneous equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0\,,$$

then a particular solution $y_p(t)$ of the 2^{nd} order nonhomogeneous linear equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = g(t) \quad (*)$$

has the form $\boxed{y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)}$, where

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}\,, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}\,.$$

⚠ These formulas are valid ONLY if the leading coefficient of y'' in $(*)$ is **1**.

Variation of Parameters (n^{th} order linear nonhomogeneous equation) :

If $y_1(t), y_2(t), \cdots, y_n(t)$ form a FSS to the n^{th} order linear homogeneous equation

$$L[y] = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0\,,$$

then a particular solution $y_p(t)$ of the n^{th} order linear nonhomogeneous equation

$$L[y] = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t) \quad (**)$$

has the form $\boxed{y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t) + \cdots + u_n(t)y_n(t)}$, where

$$u_k'(t) = \frac{\det(W_k)}{\det(W)}\,, \quad k = 1, 2, \cdots, n$$

$$W = \begin{pmatrix} y_1 & y_2 & \cdots & \mathbf{y_k} & \cdots & y_n \\ y_1' & y_2' & \cdots & \mathbf{y_k'} & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & \mathbf{y_k^{(n-1)}} & \cdots & y_n^{(n-1)} \end{pmatrix} \quad \text{and} \quad W_k = \begin{pmatrix} y_1 & y_2 & \cdots & \mathbf{0} & \cdots & y_n \\ y_1' & y_2' & \cdots & \mathbf{0} & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & \mathbf{g(t)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

⚠ These formulas are valid ONLY if the leading coefficient of $y^{(n)}$ in $(**)$ is **1**.