CHALLENGE PROBLEMS

 # 1 If $b > 0$ and $y > -\dfrac{a}{b}\,x$, find the general solution of the first order nonlinear equation

$$\frac{dy}{dx} = \sqrt{ax + by}\,.$$

Express the general solution in *implicit* form.

 # 2 The special pde $\boxed{\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0}$ is called *Laplace's Equation.*

If $f(t)$ is any arbitrary differentiable function and $a, b > 0$, show that the substitution

$$u(x, y) = e^{\frac{x}{a}}\, f(t)\,,$$

where $t = (bx - ay)$, will transform Laplace's Equation into the following (simpler) ode:

$$(a^2 + b^2) f''(t) + \frac{2b}{a}\, f'(t) + \frac{1}{a^2} f(t) = 0\,.$$

 # 3 Consider the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$. Prove that there exists an integrating factor, $\mu(y)$, depending only on y if and only if

$$\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{M(x, y)} = \omega(y)\,.$$

Moreover, $\mu(y) = e^{-\int \omega(y)\,dy}$.

 # 4 An equation of the form: $t^2\,\dfrac{d^2y}{dt^2} + \alpha t\,\dfrac{dy}{dt} + \beta\, y = 0 \quad (t > 0) \quad (*)$

is called a **Cauchy-Euler Equation** and has applications in physics.

(Note that the coefficients are functions of t.)

Use the substitution $\boxed{x = \ln t}$ to find $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$ in terms of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and then show that the equation $(*)$ becomes

$$\frac{d^2y}{dx^2} + (\alpha - 1)\,\frac{dy}{dx} + \beta\, y = 0 \quad (**)$$

(Note that this transformed equation does have constant coefficients and can be easily solved using Characteristic Roots.)

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 # 5 If $p(t)$ and $q(t)$ are continuous on an open interval I and $y_1(t)$ and $y_2(t)$ are two solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0\,,$$

prove that the Wronskian of y_1 and y_2 is given by

$$W(y_1, y_2)(t) = C\, e^{-\int p(t)\, dt}.$$

(This is known as **Abel's Theorem** and says that given any two solutions to the differential equation $L[y] = 0$ above, either $W(y_1, y_2)(t) \equiv 0$ or $W(y_1, y_2)(t) \neq 0$ for all $t \in I$. Abel's Theorem also gives us a way to compute the Wronskian $W(y_1, y_2)$, without actually knowing what y_1 and y_2 are !)

 # 6 If $x, y > 0$, find an explicit solution to

$$xy' + y \ln x = y \ln y\,.$$

 # 7 Show that $\dfrac{dy}{dx} = \dfrac{y}{x}\, F(xy)$ can be transformed into a *Separable Equation*.

 # 8 Let $p(t), q(t)$ and $g(t)$ be continuous on an open interval I containing t_0. If $y_1(t)$ and $y_2(t)$ form a FSS for $y'' + p(t)y' + q(t)y = 0$, show that a particular solution to $y'' + p(t)y' + q(t)y = g(t)$ is

$$y_p(t) = \int_{t_0}^{t} G(t, s)\, g(s)\, ds\,,$$

where

$$G(t, s) = \frac{y_1(s)\, y_2(t) - y_1(t)\, y_2(s)}{W[y_1, y_2](s)}\,.$$

The function $G(t, s)$ is called a "**Green's Function**" and is very useful in solving differential equations.

 # 9 Consider the n^{th} order linear homogeneous differential equation with constant coefficients

$$L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0\,.$$

Suppose it has a characteristic root r_1 of multiplicity $m \geq 1$. In other words, the Characteristic equation looks like

$$P(r) = (r - r_1)^m\, Q(r) = 0\,, \quad \text{where} \quad \deg Q = n - m \quad \text{and} \quad Q(r_1) \neq 0\,.$$

Show that $\displaystyle \frac{\partial^k}{\partial r^k}\Big(L[e^{rt}]\Big)\Big|_{r=r_1} = \frac{\partial^k}{\partial r^k}\Big(e^{rt} P(r)\Big)\Big|_{r=r_1} = 0\,, \quad \text{for} \quad k = 0, 1, 2, \cdots, m-1$

<u>but</u> $\displaystyle \frac{\partial^m}{\partial r^m}\Big(L[e^{rt}]\Big)\Big|_{r=r_1} \neq 0\,.$

 # 10 Consider the $n \times n$ linear system

$$\mathbf{x}' = A\mathbf{x} \quad (*)$$

where A is an $n \times n$ matrix. If $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots, \mathbf{v}^{(n)}$ are **any** set of n linearly independent vectors in $\mathbb{R}^n$, prove that the n vector functions

$$\mathbf{x}^{(k)}(t) = e^{At}\,\mathbf{v}^{(k)} \quad \text{for} \quad k = 1, 2, \cdots, n$$

is a Fundamental Set of Solutions (FSS) to $\mathbf{x}' = A\mathbf{x}$.

Note: This gives an alternative method of solving linear systems $(*)$.

 # 11 If $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$, use mathematical induction to prove that

$$A^n = \begin{pmatrix} \dfrac{3^n + (-1)^n}{2} & \dfrac{3^n - (-1)^n}{4} \\ \{3^n - (-1)^n\} & \dfrac{3^n + (-1)^n}{2} \end{pmatrix}, \quad \forall n \in \mathbb{N}.$$

 # 12 Sketch the phase portrait for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x}.$$

 # 13 .
