Solutions

MA 510 - Spring 2010

# Problem Set # 10

(due: April 16)

**1.** Page 427 : # 3(a).

**2.** Find the mass of a wire bent in the shape of a helix $\vec{c}(t) = (\cos 2t, \sin 2t, t)$, for $0 \leq t \leq \frac{\pi}{2}$, and whose density is $f(x, y, z) = 16y$.

**3.** Page 447 : # 1(d), 2(a)(d), 4(a), 7.

**4.** If $C$ is the curve $y = x^2 + 2x$ from $(0, 0)$ to $(1, 3)$, compute the following :

(a) *Path integral* : $I_1 = \int_C f\, ds$, where $f(x, y) = x^2 - y - 2$.

(b) *Line integral* : $I_2 = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$, where $\vec{\mathbf{F}} = (xy + 1)\,\vec{\mathbf{i}} - x\,\vec{\mathbf{j}}$.

(c) *Line integral* : $I_3 = \int_C 2xy\, dx + x\, dy$.

**5.** Compute the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$, where $\vec{\mathbf{F}} = (2xe^{2y} - \cos y)\,\vec{\mathbf{i}} + (2x^2e^{2y} + x \sin y - 2y)\,\vec{\mathbf{j}}$ and $C$ is *any* smooth curve starting at $(1, 0)$ and ending at $(2, 1)$.

(*Hint*: Find a function $f(x, y)$, if possible, so that $\vec{\mathbf{F}} = \nabla f$.)

$\boxed{1}$ page 427 # 3 (a): $f(x,y,z) = e^{\sqrt{z}}$

$\vec{c}(t) = (1, 2, t^2), \quad 0 \le t \le 1$

$\Rightarrow f(\vec{c}(t)) = e^{\sqrt{t^2}} = e^{|t|} = e^t \quad (\text{since } 0 \le t \le 1)$

$\therefore \int_{\vec{c}} f\, ds = \int_0^1 f(\vec{c}(t)) \|\vec{c}\,'(t)\| dt = \int_0^1 e^t \|(0,0,2t)\| dt$

$= \int_0^1 2te^t\, dt = \left(2te^t - 2e^t\right)\Big|_{t=0}^{1} = (2e - 2e) - (0 - 2) = \underline{\underline{2}}$

$\boxed{2}$ $\vec{c}(t) = (\cos 2t, \sin 2t, t), \quad 0 \le t \le \frac{\pi}{2}$

$f(x,y,z) = 16y \Rightarrow f(\vec{c}(t)) = 16 \sin 2t$

$\vec{c}\,'(t) = (-2\sin 2t, 2\cos 2t, 1)$

$\therefore \text{Mass} = \int_{\vec{c}} f\, ds = \int_0^{\frac{\pi}{2}} f(\vec{c}(t)) \|\vec{c}\,'(t)\| dt$

$= \int_0^{\frac{\pi}{2}} (16 \sin 2t)\sqrt{5}\, dt = \underline{\underline{16\sqrt{5}}}$

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page 447 #1(d): $\vec{F}(x,y,z) = (x,y,z)$

$\vec{c}(t) = (t^2, 3t, 2t^3)$, $-1 \le t \le 2$

$\Rightarrow \vec{F}(\vec{c}(t)) = (t^2, 3t, 2t^3)$ and $\vec{c}'(t) = (2t, 3, 6t^2)$

$$\therefore \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{-1}^{2} \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)\,dt$$

$$= \int_{-1}^{2} (t^2, 3t, 2t^3) \cdot (2t, 3, 6t^2)\,dt = \int_{-1}^{2} (2t^3 + 9t + 12t^5)\,dt$$

$$= \underline{\underline{147}}$$

page 447 #2(a): $\vec{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

$\Rightarrow$ $x = \cos t$

$y = \sin t$

$$\therefore \int_{\vec{c}} x\,dy - y\,dx = \int_0^{2\pi} (\cos t)(\cos t\,dt) - (\sin t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} dt$$

$$= \underline{\underline{2\pi}}$$

**page 447 # 2(d):**

$\vec{c}$ is that parabola $z = x^2,\ y=0$ from $(-1,0,1)$ to $(1,0,1)$

We can parametrize $\vec{c}$ if we let $\begin{cases} x = t \\ y = 0 \\ z = t^2 \end{cases}$ $\quad -1 \le t \le 1$

$$\therefore \int_{\vec{c}} x^2\,dx - xy\,dy + dz = \int_{-1}^{1} t^2\,dt - 0 + 2t\,dt = \underline{\underline{\frac{2}{3}}}$$

**page 447 # 4(a):** Since $\vec{F} \perp \vec{c}\,'$ @ $\vec{c}(t) \Rightarrow \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t) = 0$

Hence $$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt = \int_a^b 0\,dt = 0.$$

**page 447 # 7:** $\vec{c}(t) = (t, t^n, 0),\quad 0 \le t \le 1$
$n = 1, 2, 3, \ldots$

hence $\begin{array}{l} x = t \\ y = t^n \\ z = 0 \end{array}$

so $$\int_{\vec{c}} y\,dx + (3y^3 - x)\,dy + z\,dz = \int_0^1 t^n(dt) + (3t^{3n} - t)(nt^{n-1}\,dt) + 0$$

$$= \int_0^1 \{t^n + 3nt^{4n-1} - nt^n\}\,dt = \boxed{\frac{3}{4} + \frac{1-n}{n+1}}$$

$\boxed{4}$ Parametrize $C$ by $\begin{cases} x = t \\ y = t^2 + 2t \end{cases}$, $0 \le t \le 1$

i.e., $\vec{c}(t) = (t,\ t^2+2t)$, $0 \le t \le 1$

so $\vec{c}\,'(t) = (1,\ 2t+2)$

Hence we get

(a) $I_1 = \int_C (x^2 - y - 2)\,ds = \int_0^1 \{t^2 - (t^2+2t) - 2\}\,\|(1,\ 2t+2)\|\,dt$

$$= \int_0^1 -(2t+2)\sqrt{1+(2t+2)^2}\,dt = \int_2^4 -u\sqrt{1+u^2}\left(\tfrac{1}{2}\,du\right) = \frac{5^{3/2} - 17^{3/2}}{6}$$

(b) $I_2 = \int_C \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$

$$= \int_0^1 (t^3 + 2t^2 + 1,\ -t) \cdot (1,\ 2t+2)\,dt = \int_0^1 (t^3 - 2t + 1)\,dt = \frac{1}{4}$$

(c) $I_3 = \int_C 2xy\,dx + x\,dy = \int_0^1 2(t)(t^2+2t)\,dt + t(2t+2)\,dt$

$$= \int_0^1 (2t^3 + 6t^2 + 2t)\,dt = \frac{7}{2}$$

$\boxed{5}$ $\vec{F}(x,y) = \left(2xe^{2y} - \cos y \, , \, 2x^2e^{2y} + x\sin y - 2y\right) = \nabla f$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 2xe^{2y} - \cos y \xrightarrow{I_x} f = x^2e^{2y} - x\cos y + g(y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow D_y \\ \dfrac{\partial f}{\partial y} = 2x^2e^{2y} + x\sin y - 2y = \dfrac{\partial f}{\partial y} = 2x^2e^{2y} + x\sin y + g'(y) \end{cases}$$

$$g'(y) = -2y \Rightarrow g(y) = -y^2 + C$$

$\therefore$ $\vec{F}(x,y) = \nabla f(x,y)$, where $f(x,y) = x^2e^{2y} - x\cos y - y^2 + C$

Hence by FTC for line integrals,

$$\int_C \vec{F}\cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} = f(2,1) - f(1,0)$$

$$= \{4e^2 - 2\cos 1 - 1 + c\} - \{1 - 1 + c\}$$

$$= \underline{\underline{4e^2 - 2\cos 1 - 1}}$$