Solutions

MA 510 - Spring 2010

Problem Set # 11

(due: April 23)

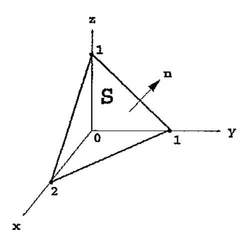
1. Page 459 : # 10, 12.

2. Compute the area of the surface S parameterized by $\Phi(u,v) = (u\cos v,\, u\sin v,\, v)$ where $0 \le u \le \sqrt{8},\ 0 \le v \le u$.

3. Parameterize the surface S given by $y = x^2 + z^2 - 8$ where $1 \le x^2 + z^2 \le 4$. Find the area of the surface S.

4. Page 480 : # 3, 10.

5. Compute the surface integrals $\iint_S x\, dS$ and $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$, where $\vec{\mathbf{F}}(x,y,z) = (z,\, 4x,\, 2y+1)$ and S is that part of the plane $\frac{x}{2} + y + z = 1$ in the 1^{st} octant:



6. Compute the surface integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}}(x,y,z) = y\vec{\mathbf{i}} - x\vec{\mathbf{j}} + z\vec{\mathbf{k}}$ and S is that part of the paraboloid $z = 9 - x^2 - y^2$ which lies above the plane $z = 5$ and $\vec{\mathbf{N}}$ is the upward unit normal. What is the *flux* of $\vec{\mathbf{F}}$ across S ?

$\boxed{1}$

page 459 #10: $\Phi(u,v) = (u^2, v^2, u^2+v^2)$

$x = u^2$
$y = v^2$
$z = u^2+v^2$

$$\vec{n} = \vec{T}_u \times \vec{T}_v = (2u, 0, 2u) \times (0, 2v, 2v) = (-4uv, -4uv, 4uv)$$

Now $u_0 = 1$, $v_0 = 1$ $\Rightarrow$ $(x_0, y_0, z_0) = (1, 1, 2)$

and $\vec{n} = (-4, -4, 4)$

$\therefore$ Equation of tangent plane becomes

$$(x-1, y-1, z-2) \cdot (-4, -4, 4) = 0$$

$$-4(x-1) - 4(y-1) + 4(z-2) = 0$$

or $$-(x-1) - (y-1) + (z-2) = 0$$

$$\underline{-x - y + z = 0}$$

page 459 #12: Surface is $x^3 + 3xy + z^2 = 2$ $(z>0)$
and $(x_0, y_0, z_0) = (1, \frac{1}{3}, 0)$

Let $x = u$, $z = v$ then since $x^3 + 3xy + z^2 = 2$

$$\Rightarrow u^3 + 3uy + v^2 = 2 \Rightarrow y = \frac{2 - v^2 - u^3}{3u}$$

$\therefore$ A parameterization is

$$\begin{cases} x = u \\ y = \dfrac{2 - v^2 - u^3}{3u} \\ z = v \end{cases}$$

$\vec{n} = \vec{T}_u \times \vec{T}_v$, where

$$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \left(1, \frac{v^2 - 2}{3u^2} - \frac{2u}{3}, 0\right)$$

$$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(0, -\frac{2v}{3u}, 1\right)$$

Now $(u_0, v_0) = (x_0, z_0) = (1, 0)$

$$\therefore \vec{n} = (1, -\tfrac{4}{3}, 0) \times (0, 0, 1) = (-\tfrac{4}{3}, -1, 0)$$

So equation of tangent plane is $(x-1, y-\frac{1}{3}, z-0) \cdot (-\frac{4}{3}, -1, 0) = 0$

$$\Rightarrow \underline{4x + 3y = 5}$$

(cont'd)

Using level sets: let $F(x,y,z) = x^3+3xy+z^2 = 2$

Then a normal vector @ $(1, \frac{1}{3}, 0)$ is

$$\vec{n} = \nabla F(1, \tfrac{1}{3}, 0) = (3x^2+3y, 3x, 2z)\Big|_{(1,\frac{1}{3},0)} = (4,3,0)$$

so equation of tangent plane is

$$(x-1, y-\tfrac{1}{3}, z)\cdot(4,3,0) = 0$$

$$4(x-1)+3(y-\tfrac{1}{3}) = 0$$

$$\underline{\underline{4x+3y=5}}$$

$\boxed{2}$ $\Phi(u,v) = (u\cos v, u\sin v, v) \Rightarrow \begin{cases} x = u\cos v \\ y = u\sin v \\ z = v \end{cases}$

$D: \; 0 \le u \le \sqrt{8}, \; 0 \le v \le u$

$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos v, \sin v, 0)$

$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-u\sin v, u\cos v, 1)$

$\therefore \; \vec{T}_u \times \vec{T}_v = (\sin v, -\cos v, u) \Rightarrow \|\vec{T}_u \times \vec{T}_v\| = \sqrt{1+u^2}$

$\therefore \; A(S) = \iint_D \|\vec{T}_u \times \vec{T}_v\| \, du\, dv = \int_0^{\sqrt{8}} \int_0^{u} \sqrt{1+u^2}\, dv\, du = \underline{\underline{\frac{26}{3}}}$

$\boxed{3}$ Let $S: \begin{cases} x = r\cos\theta \\ y = r^2 - 8 \\ z = r\sin\theta \end{cases}$ and $D: \; 1 \le r \le 2, \; 0 \le \theta \le 2\pi$

$\therefore \; \Phi(r,\theta) = (r\cos\theta, r^2-8, r\sin\theta)$ where $D: \; 1 \le r \le 2, \; 0 \le \theta \le 2\pi$

(one of many possibilities)

$\vec{T}_r = \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right) = (\cos\theta, 2r, \sin\theta)$

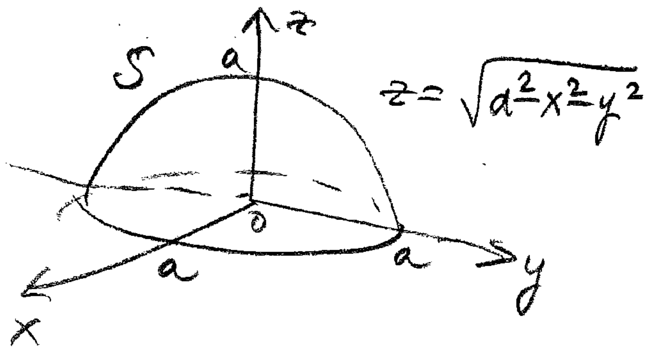
$\vec{T}_\theta = \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) = (-r\sin\theta, 0, r\cos\theta)$

$\Rightarrow \|\vec{T}_r \times \vec{T}_\theta\| = r\sqrt{4r^2+1}$

$\therefore \; A(S) = \iint_D \|\vec{T}_r \times \vec{T}_\theta\| \, dr\, d\theta = \int_0^{2\pi} \int_1^{2} r\sqrt{4r^2+1}\, dr\, d\theta = \underline{\underline{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}}$

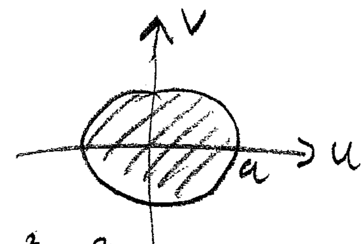
$\boxed{4}$

page 480 #3:



Soln 1 (Rectangular Coordinates)

Let $S: \begin{cases} x = u \\ y = v \\ z = \sqrt{a^2-u^2-v^2} \end{cases}$ where D:



$u^2+v^2 \le a^2$

$$\vec{T}_u = \left(1, 0, \frac{-u}{\sqrt{a^2-u^2-v^2}}\right), \quad \vec{T}_v = \left(0, 1, \frac{-v}{\sqrt{a^2-u^2-v^2}}\right)$$

$$\Rightarrow \|\vec{T}_u \times \vec{T}_v\| = \left\|\left(\frac{u}{\sqrt{a^2-u^2-v^2}}, \frac{v}{\sqrt{a^2-u^2-v^2}}, 1\right)\right\| = \frac{a}{\sqrt{a^2-u^2-v^2}}$$

$$\therefore \iint_S z\,dS = \iint_D \left(\sqrt{a^2-u^2-v^2}\right)\left(\frac{a}{\sqrt{a^2-u^2-v^2}}\right) du\,dv = a\iint_D du\,dv$$

$$= a\left(\pi a^2\right) = \underline{\underline{\pi a^3}}$$

(cont'd)

Soln 2 (Cylindrical Coordinates)

Let $S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{a^2 - r^2} \end{cases}$ where $D: \begin{array}{l} 0 \le r \le a \\ 0 \le \theta \le 2\pi \end{array}$

$\vec{T}_r = \left(\cos\theta, \sin\theta, \frac{-r}{\sqrt{a^2-r^2}}\right)$; $\vec{T}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\Rightarrow \|\vec{T}_r \times \vec{T}_\theta\| = \dfrac{ar}{\sqrt{a^2-r^2}}$

$$\therefore \iint_S z\,dS = \iint_D \left(\sqrt{a^2-r^2}\right)\left(\frac{ar}{\sqrt{a^2-r^2}}\right) dr\,d\theta$$

$$= \iint_D a r\,dr\,d\theta = \int_0^{2\pi}\int_0^a a r\,dr\,d\theta = \underline{\underline{\pi a^3}}$$

Soln 3 (Spherical Coordinates)

Let $S: \begin{cases} x = (a\sin\phi)\cos\theta \\ y = (a\sin\phi)\sin\theta \\ z = a\cos\phi \end{cases}$, $D: \begin{cases} 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{cases}$

$\Rightarrow \vec{T}_\phi = (a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi)$

$\vec{T}_\theta = (-a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0)$

(cont'd)

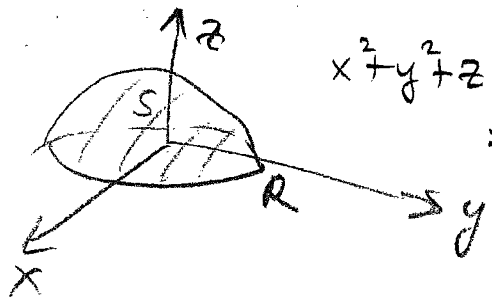
$$\vec{T}_\phi \times \vec{T}_\theta = a^2 \sin\phi \left(\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \right)$$

$$\Rightarrow \|\vec{T}_\phi \times \vec{T}_\theta\| = a^2 \sin\phi$$

$$\therefore \iint_S z \, dS = \iint_D a \cos\phi \left(a^2 \sin\phi\right) d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a^3 \cos\phi \sin\phi \; d\phi \; d\theta = \underline{\underline{\pi a^3}}$$

Page 481 #10 :



$x^2+y^2+z^2=R^2$

$z \geq 0$

(8)

$S: \begin{cases} x = (R\sin\phi)\cos\theta \\ y = (R\sin\phi)\sin\theta \\ z = R\cos\phi \end{cases}$ where $D: \begin{array}{l} 0 \leq \phi \leq \frac{\pi}{2} \\ 0 \leq \theta \leq 2\pi \end{array}$

$\vec{T}_\phi \times \vec{T}_\theta = (R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi) \times (-R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0)$

$\Rightarrow \vec{T}_\phi \times \vec{T}_\theta = R^2\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$\therefore \|\vec{T}_\phi \times \vec{T}_\theta\| = R^2\sin\phi$

Mass density $m(x,y,z) = x^2+y^2 = R^2\sin^2\phi$

$\therefore$ Mass $= \iint_S m\,dS = \iint_D (R^2\sin^2\phi)\|\vec{T}_\phi \times \vec{T}_\theta\|\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (R^2\sin^2\phi)(R^2\sin\phi)\,d\phi\,d\theta$

$= 2\pi R^4 \int_0^{\frac{\pi}{2}} \sin^3\phi\,d\phi = 2\pi R^4\left(\frac{2}{3}\right) = \frac{4\pi R^4}{3}$

9

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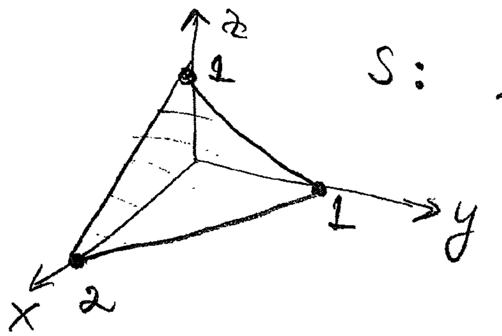
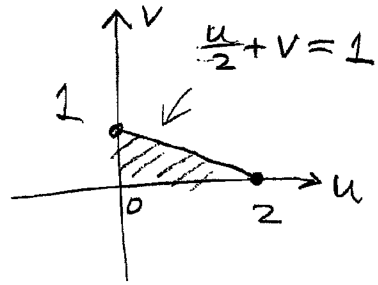


$S: \quad \frac{x}{2} + y + z = 1$

Parametrize $S: \begin{cases} x = u \\ y = v \\ z = 1 - \frac{u}{2} - v \end{cases}$ where D:



$$\vec{T}_u \times \vec{T}_v = \left(1, 0, -\tfrac{1}{2}\right) \times (0, 1, -1) = \left(\tfrac{1}{2}, 1, 1\right)$$

and $\|\vec{T}_u \times \vec{T}_v\| = \frac{3}{2}$

$$\therefore \iint_S x\, dS = \iint_D u\left(\tfrac{3}{2}\right) du\, dv = \int_0^2 \int_0^{1-\frac{u}{2}} u\left(\tfrac{3}{2}\right) dv\, du = \underline{\underline{1}}$$

$\vec{F}(x,y,z) = (z, 4x, 2y+1)$

And so

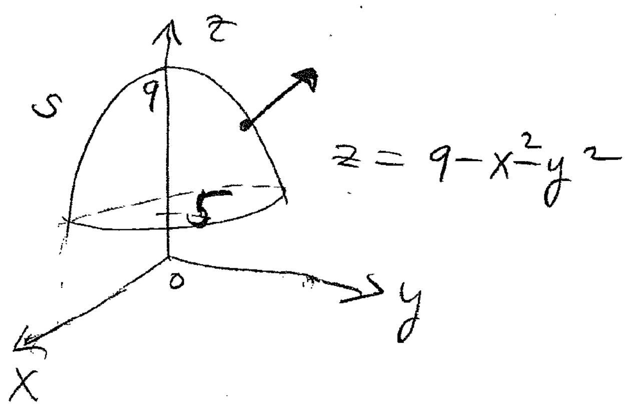
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(1 - \tfrac{u}{2} - v, 4u, 2v+1\right) \cdot \left(\tfrac{1}{2}, 1, 1\right) du\, dv$$

$$= \iint_D \left(\tfrac{1}{2} - \tfrac{u}{4} - \tfrac{v}{2} + 4u + 2v + 1\right) du\, dv$$

$$= \iint_D \left(\tfrac{3}{2} + \tfrac{15u}{4} + \tfrac{3v}{2}\right) du\, dv$$

$$= \int_0^2 \int_0^{1-\frac{u}{2}} \left(\tfrac{3}{2} + \tfrac{15u}{4} + \tfrac{3v}{2}\right) dv\, du = \underline{\underline{\tfrac{9}{2}}}$$

$\boxed{6}$ (10)



Parametrize S: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 9 - r^2 \end{cases}$, where D: $0 \le r \le 2$, $0 \le \theta \le 2\pi$

$$\vec{T}_r \times \vec{T}_\theta = (\cos\theta, \sin\theta, -2r) \times (-r\sin\theta, r\cos\theta, 0)$$

$$\Rightarrow \vec{T}_r \times \vec{T}_\theta = (2r^2\cos\theta, 2r^2\sin\theta, r)$$ points in correct direction since z component > 0

$\|\vec{T}_r \times \vec{T}_\theta\| = r\sqrt{4r^2+1}$, $\vec{F}(x,y,z) = (y, -x, z)$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iint_D (r\sin\theta, -r\cos\theta, 9-r^2)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)\,dr\,d\theta$$

$$= \iint_D (9r - r^3)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (9r - r^3)\,dr\,d\theta = \underline{\underline{28\pi}}$$

Since $\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\vec{N})\,dS = \underline{\underline{28\pi}}$ = flux across S