Solutions

MA 510 - Spring 2010

Problem Set # 12 (Optional)

(due: April 30)

1. Page 178 : # 37(b).

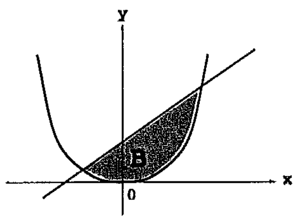
2. Page 243 : # 5.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, prove that $\nabla(fg) = f\nabla g + g\nabla f$.

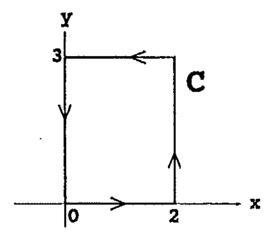
4. If $g(x, y, z) = (x,\ x + y,\ x^2 + z,\ z)$ and $f(x_1, x_2, x_3, x_4) = (x_1^3 + x_3, x_2^2 - x_4)$, then compute $D(f \circ g)(1, 1, 1)$.

5. If $z = z(x, y)$ is defined implicitly by the equation $\boldsymbol{x^2 + z^3 + 3xy - 3z = 1}$, compute $\dfrac{\partial z}{\partial y}$.

6. The base of a solid S lies in the region B between $y = x^2$ and $y = x + 2$ in the xy-plane as shown below. Plane sections perpendicular to the x-axis are squares with one side in the xy-plane. Find the volume of S.



7. Compute $\displaystyle\int_C -5xy\, dy + (x^3 + \cos^2 x - 4y)\, dx$ where C is as shown:



8. If $\vec{\mathbf{F}}(x, y, z) = -yz\,\vec{\mathbf{i}} + xz\,\vec{\mathbf{j}} + y\,\vec{\mathbf{k}}$ and S is that part of $z = x^2 + y^2$ below $z = 4$ (the normal $\vec{\mathbf{n}}$ points downward), compute $\displaystyle\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ using Stokes' Theorem.

9. Let $\vec{\mathbf{F}}(x, y, z) = (x, y, -3z)$.

(a) Compute $\displaystyle\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where S_1 is that part of $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$ and $\vec{\mathbf{n}}$ is downward.

(b) Compute $\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$, where S is the *closed* surface consisting of that part of $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$ together with the top and $\vec{\mathbf{n}}$ points outward.

10. Page 574 : # 10.

1 **page 178 #37(b):** $z = \frac{u^2+v^2}{u^2-v^2}$ where $u = e^{-x-y}$, $v = e^{xy}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \left(\frac{-4uv^2}{(u^2-v^2)^2}\right)\left(-e^{-x-y}\right) + \left(\frac{4u^2v}{(u^2-v^2)^2}\right)\left(ye^{xy}\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \left(\frac{-4uv^2}{(u^2-v^2)^2}\right)\left(-e^{-x-y}\right) + \left(\frac{4u^2v}{(u^2-v^2)^2}\right)\left(xe^{xy}\right)$$

2 **page 243 #5:** Extremize $f(x,y) = 3x+2y$
s.t. $g(x,y) = 2x^2+3y^2 = 3$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 3 \end{cases} \Rightarrow \begin{cases} \langle 3, 2\rangle = \lambda \langle 4x, 6y\rangle \\ 2x^2+3y^2 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} 3 = 4\lambda x & (1) \\ 2 = 6\lambda y & (2) \\ 2x^2+3y^2 = 3 & (3) \end{cases}$$

$(1) \Rightarrow \lambda = \frac{3}{4x}$ $\therefore (2) \Rightarrow 2 = 6\left(\frac{3}{4x}\right)y$

$\therefore \quad x = \frac{9}{4}y$

Now $(3) \Rightarrow 2\left(\frac{9}{4}y\right)^2 + 3y^2 = 3 \Rightarrow y = \pm\sqrt{\frac{8}{35}} = \pm\frac{4}{\sqrt{70}}$

hence, $x = \pm\frac{9}{4}\sqrt{\frac{8}{35}} = \pm\frac{9}{\sqrt{70}}$

(x, y)	$f(x,y) = 3x+2y$	
$\left(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}}\right)$	$-\frac{35}{\sqrt{70}}$	$\leftarrow$ min value
$\left(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}}\right)$	$\frac{35}{\sqrt{70}}$	$\leftarrow$ max value

[3] $f, g : \mathbb{R}^2 \to \mathbb{R}$

$$\nabla(fg) = \left\langle \frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y} \right\rangle = \left\langle f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y} \right\rangle$$

$$= f\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle + g\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = f\nabla g + g\nabla f \quad \blacksquare$$

[4] $g : \mathbb{R}^3 \to \mathbb{R}^4, \; g(x,y,z) = (x, x+y, x^2+z, z)$

$f : \mathbb{R}^4 \to \mathbb{R}^2, \; f(x_1, x_2, x_3, x_4) = (x_1^3 + x_3, x_2^2 - x_4)$

Now $Dg = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2x & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $Df = \begin{bmatrix} 3x_1^2 & 0 & 1 & 0 \\ 0 & 2x_2 & 0 & -1 \end{bmatrix}$

$g(1,1,1) = (1,2,2,1)$

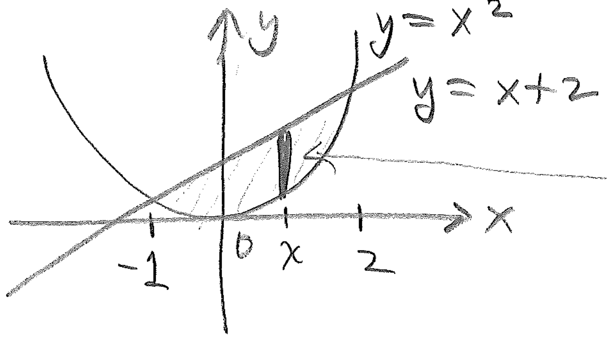
$$\therefore D(f\circ g)(1,1,1) = Df(1,2,2,1)\, Dg(1,1,1) = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 5 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix}}}$$

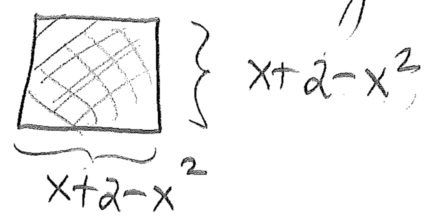
[5] Let $F(x,y,z) = x^2 + z^3 + 3xy - 3z - 1$. Then $z = z(x,y)$ is defined implicitly by $F(x,y,z) = 0$.

Thus, $$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \boxed{-\frac{3x}{3z^2 - 3}}$$

6 (3)



cross sections are squares:



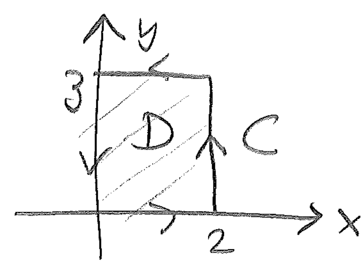
$x^2 = x+2 \Rightarrow x^2 - x - 2 = (x-2)(x+1) = 0$

$\therefore$ cross-sectional area @ x is $A(x) = [x+2-x^2]^2$

By Cavalieri's Principle Volume $= \int_a^b A(x)\,dx$

so $V = \int_{-1}^{2} [x+2-x^2]^2 dx = \boxed{\frac{81}{10}}$

7 $\int_C \overset{Q}{-5xy}\,dy + (\overset{P}{x^3 + \cos^2 x - 4y})\,dx$

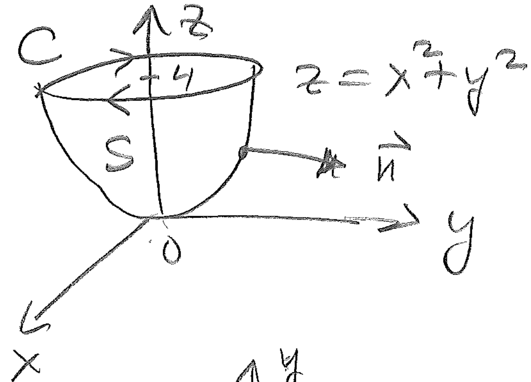


$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ (by Green's Theorem)

$= \iint_D (-5y + 4)\,dx\,dy = -5 \iint_D y\,dx\,dy + 4 \iint_D dx\,dy$

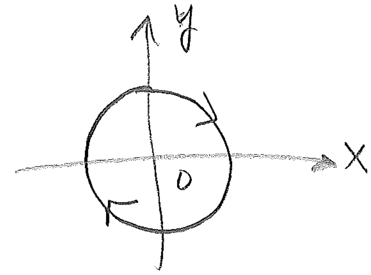
$= -5\bar{y}A + 4A = -5\left(\frac{3}{2}\right)(6) + 4(6) = \boxed{-21}$

8



$\vec{F}(x,y,z) = (-yz, xz, y)$

Hence $C = \partial S$



and $z = 4$

A parametrization for $-C$ is $\vec{c}(t) = (2\cos t, 2\sin t, 4)$

$0 \le t \le 2\pi$

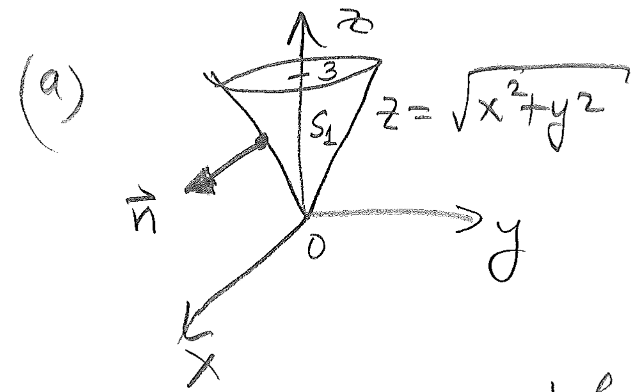
Stokes' Thm $\Rightarrow$ $$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{s} = -\int_{-C} \vec{F} \cdot d\vec{s}$$

$$= -\int_0^{2\pi} (-8\sin t, 8\cos t, 2\sin t) \cdot (-2\sin t, 2\cos t, 0)\, dt$$

$$= -\int_0^{2\pi} 16\, dt = \boxed{-32\pi}$$

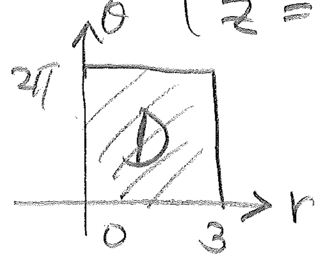
9 $\vec{F}(x,y,z) = (x, y, -3z)$

(a)



$\Rightarrow S_1: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases}$

where



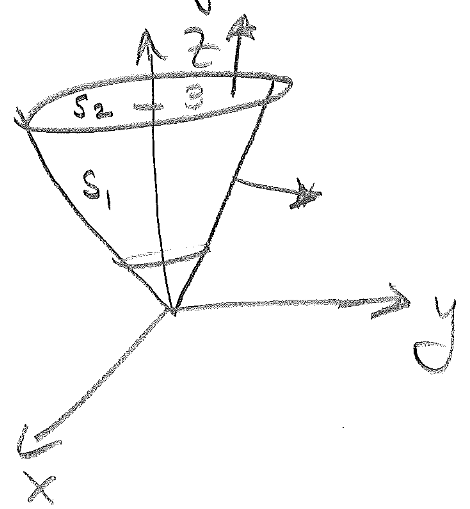
$\vec{T}_r \times \vec{T}_\theta = (-r\cos\theta, -r\sin\theta, r)$

WRONG direction for normal (hence this parametrization is orientation reversing) so

Thus, $\displaystyle\iint_{S_1} \vec{F} \cdot d\vec{S} = -\iint_D (r\cos\theta, r\sin\theta, -3r) \cdot (-r\cos\theta, -r\sin\theta, r)\, dr\, d\theta$

$\displaystyle = -\int_0^{2\pi}\int_0^3 -4r^2\, dr\, d\theta = \boxed{72\pi}$

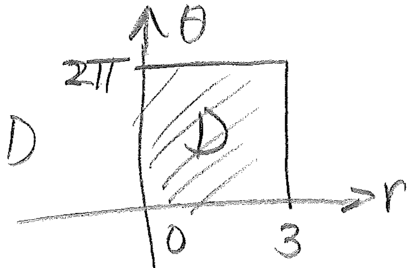
9 (b) $\vec{F}(x,y,z) = (x, y, -3z)$



$S = S_1 \cup S_2$

where S_2 is the disk $x^2+y^2 \le 9$, $z=3$

$\therefore\ S_2 : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 3 \end{cases}$ where $(r,\theta) \in D$



$\vec{T}_r \times \vec{T}_\theta = (0,0,r)$ correct direction of normal

$\therefore \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D (r\cos\theta, r\sin\theta, -9) \cdot (0,0,r)\, dr\, d\theta$

$= \iint_D -9r\, dr\, d\theta = -81\pi$

$\therefore$ From (a), $\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

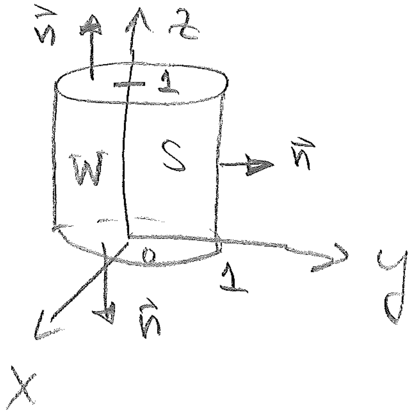
$= 72\pi - 81\pi = \boxed{-9\pi}$

Or just use Divergence Theorem since surface is closed: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} \vec{F}\, dx\,dy\,dz = \iiint_W -1\, dx\,dy\,dz = \boxed{-9\pi}$

$\boxed{10}$

page 574 #10

$\vec{F}(x,y,z) = \langle 1, 1, z(x^2+y^2)^2 \rangle$



$S = \partial W$

$$\iint_S (\vec{F}\cdot\vec{n})\,dS = \iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}\vec{F}\;dx\,dy\,dz$$

$$= \iiint_W (x^2+y^2)^2\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^1\int_0^1 r^4\;r\,dz\,dr\,d\theta$$

$$= \boxed{\frac{\pi}{3}}$$