Solutions

MA 510 - Spring 2010

Problem Set # 8

(due: March 26)

1. **Page 339** (§5.2): # 2, 4, 5.

2. **Page 347** (§5.3) : # 2(a,b,c), 4, 6, 8.

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Problem Set # 8

1 page 339 #2: $R = [0,1]\times[0,1]$

(a) $\iint_R x^m y^n \, dx\, dy = \int_0^1 \int_0^1 x^m y^n \, dx\, dy = \int_0^1 \frac{x^{m+1}}{m+1} y^n \Big|_{x=0}^{1} dy$

$= \int_0^1 \frac{1}{m+1} y^n \, dy = \boxed{\frac{1}{(m+1)(n+1)}}$

(b) $\iint_R (ax+by+c)\, dx dy = \int_0^1 \int_0^1 (ax+by+c)\, dx\, dy$

$= \int_0^1 \left(\frac{ax^2}{2} + bxy + cx\right)\Big|_{x=0}^{1} dy = \int_0^1 \left(\frac{a}{2} + by + c\right) dy = \boxed{\frac{a}{2} + \frac{b}{2} + c}$

(c) $\iint_R \sin(x+y)\, dx\, dy = \int_0^1 \int_0^1 \sin(x+y)\, dx dy = \int_0^1 -\cos(x+y)\Big|_{x=0}^{1} dx$

$= \int_0^1 \{\cos y - \cos(1+y)\} dy = \boxed{2 \sin 1 - \sin 2}$

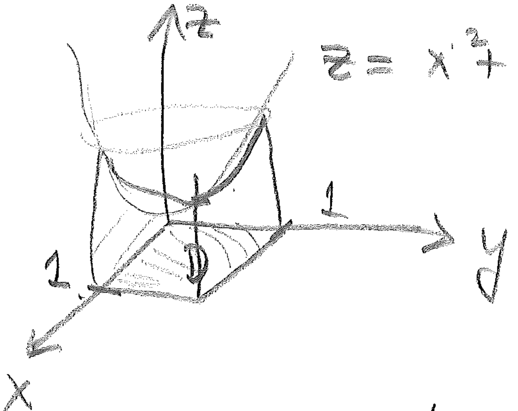
(d) $\iint_R (x^2 + 2xy + y x^{\frac{1}{2}})\, dx dy = \int_0^1 \int_0^1 (x^2 + 2xy + y x^{\frac{1}{2}})\, dx\, dy$

$= \int_0^1 \left(\frac{x^3}{3} + x^2 y + \frac{2}{3} y x^{\frac{3}{2}}\right)\Big|_{x=0}^{1} dy = \int_0^1 \left(\frac{1}{3} + y + \frac{2}{3} y\right) dy$

$= \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \boxed{\frac{7}{6}}$

page 339 #4:

$D = [0,1] \times [0,1]$



$z = x^2 + y^4$

$$V = \iint_D (x^2+y^4)\, dA = \int_0^1 \int_0^1 (x^2+y^4)\, dy\, dx = \int_0^1 \left(x^2 y + \frac{y^5}{5}\right)\Big|_{y=0}^{1} dx$$

$$= \int_0^1 \left(x^2 + \frac{1}{5}\right) dx = \frac{1}{3} + \frac{1}{5} = \boxed{\frac{8}{15}}$$

page 339 #5:

f continuous on $[a,b]$

g continuous on $[c,d]$

$R = [a,b] \times [c,d]$

$$\iint_R [f(x)g(y)]\, dx\, dy = \int_a^b \left(\int_c^d f(x)\, g(y)\, dy \right) dx$$

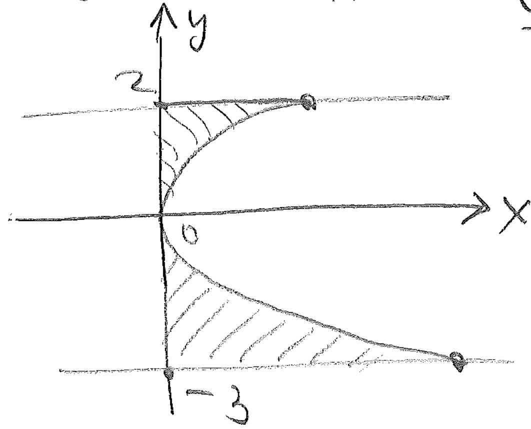
$$= \int_a^b f(x) \left(\int_c^d g(y)\, dy \right) dx$$

$$= \left(\int_a^b f(x)\, dx \right) \left(\int_c^d g(y)\, dy \right) \quad \blacksquare$$

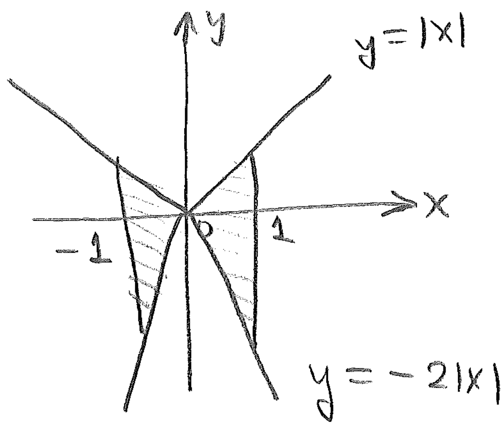
2 Page 347 # 2(a,b,c):

(a) $\int_{-3}^{2}\int_{0}^{y^2}(x^2+y)\,dx\,dy = \int_{-3}^{2}\left(\frac{x^3}{3}+xy\right)\Big|_{x=0}^{y^2} dy = \int_{-3}^{2}\left(\frac{y^6}{3}+y^3\right) dy$

$= \boxed{\frac{7895}{84}}$



(b) $\int_{-1}^{1}\int_{-2|x|}^{|x|} e^{x+y}\,dy\,dx = \underbrace{\int_{0}^{1}\int_{-2x}^{x} e^{x+y}\,dy\,dx}_{I_1} + \underbrace{\int_{-1}^{0}\int_{2x}^{-x} e^{x+y}\,dy\,dx}_{I_2}$



Now $I_1 = \int_0^1 e^{x+y}\Big|_{y=-2x}^{x} dx$

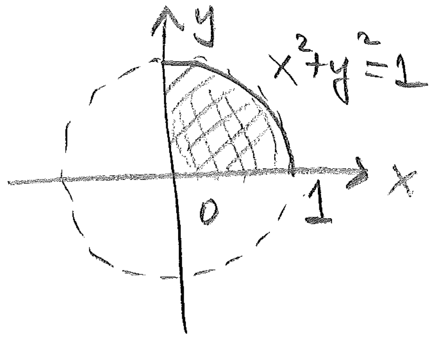
$= \int_0^1 (e^{2x} - e^{-x})\,dx = \frac{e^2}{2} + e^{-1} - \frac{3}{2}$

Similarly, $I_2 = \frac{2}{3} + \frac{e^{-3}}{3}$

$\therefore \int_{-1}^{1}\int_{-2|x|}^{|x|} e^{x+y}\,dy\,dx = I_1 + I_2 = \boxed{-\frac{5}{6} + e^{-1} + \frac{e^2}{2} + \frac{e^{-3}}{3}}$

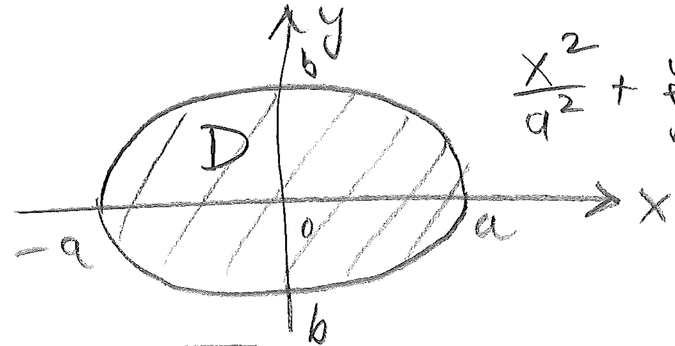
Page 347 # 2 (c):

$$\int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4} \text{ area of circle of radius } 1$$



$$= \frac{1}{4}\left(\pi (1)^2\right) = \boxed{\frac{\pi}{4}}$$

Page 347 # 4:



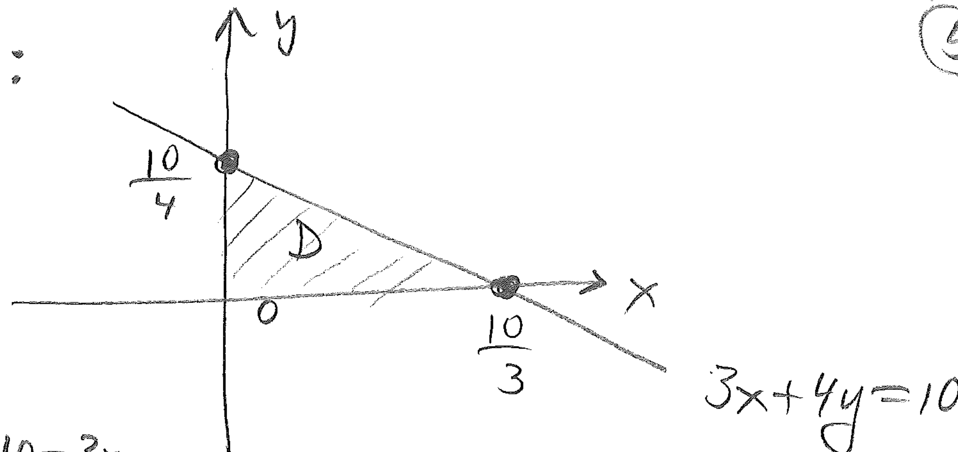
$$\text{Area} = \iint_D dx\,dy = \int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$

$$= \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\,dx = \int_{-a}^{a} \frac{2b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{2b}{a}\int_{-a}^{a} \sqrt{a^2-x^2}\,dx = \left(\frac{2b}{a}\right)\left(\frac{1}{2}\right)(\text{area of circle of radius } a)$$

$$= \left(\frac{2b}{a}\right)\left(\frac{1}{2}\right)\left(\pi a^2\right) = \boxed{\pi a b}$$

Page 347 #6:



$$D: \begin{cases} 0 \le y \le \frac{10-3x}{4} \\ 0 \le x \le \frac{10}{3} \end{cases}$$

$$\therefore \iint_D (x^2+y^2)\,dA = \int_0^{\frac{10}{3}} \int_0^{\frac{10-3x}{4}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^{\frac{10}{3}} \left(x^2y + \frac{y^3}{3}\right)\Big|_{y=0}^{y=\frac{10-3x}{4}} dx = \int_0^{\frac{10}{3}} \left[x^2\left(\frac{10-3x}{4}\right) + \frac{1}{3}\left(\frac{10-3x}{4}\right)^3\right] dx$$

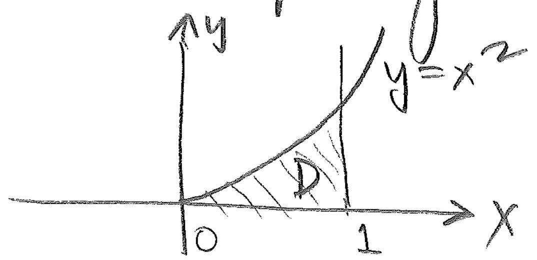
$$= \boxed{\frac{5}{24}\left(\frac{10}{3}\right)^3 + \frac{1}{9}\left(\frac{5}{2}\right)^4} = \boxed{\frac{15625}{1296}}$$

Page 347 # 8: $\int_0^1 \int_0^{x^2} (x^2+xy-y^2)\,dy\,dx = \int_0^1 \left(x^2y + \frac{xy^2}{2} - \frac{y^3}{3}\right)\Big|_{y=0}^{y=x^2} dx$

$$= \int_0^1 \left(x^4 + \frac{x^5}{2} - \frac{x^6}{3}\right) dx = \boxed{\frac{33}{140}}$$

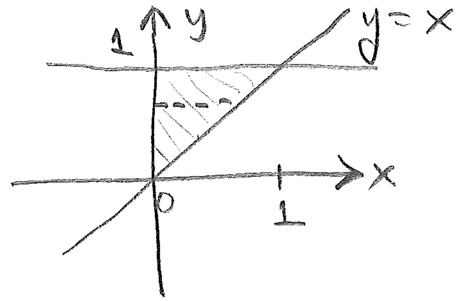
This is the double integral of $f(x,y) = x^2+xy-y^2$

over D:



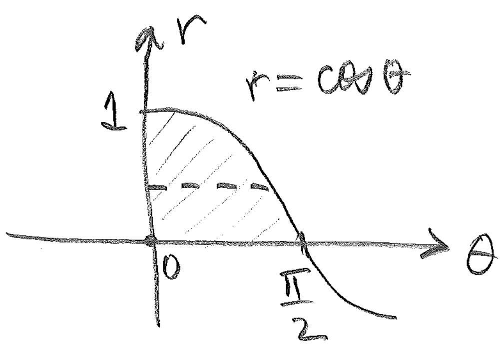
3 page 353 # 1(a, b, c, d)

(a) $\displaystyle\int_0^1\int_x^1 xy\,dy\,dx = \int_0^1\int_0^y xy\,dx\,dy = \int_0^1 \frac{x^2}{2}y\Big|_{x=0}^{y} dy$



$\displaystyle = \int_0^1 \frac{y^3}{2}\,dy = \boxed{\frac{1}{8}}$

(b) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} \cos\theta\,dr\,d\theta = \int_0^1\int_0^{\cos^{-1}(r)} \cos\theta\,d\theta\,dr$



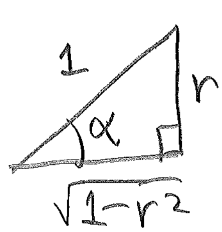
$\displaystyle = \int_0^1 \sin\theta\Big|_{\theta=0}^{\cos^{-1}(r)} dr$

$\displaystyle = \int_0^1 \sin(\cos^{-1}(r))\,dr$

Now if $\alpha = \cos^{-1} r$

then $\cos\alpha = r$

so



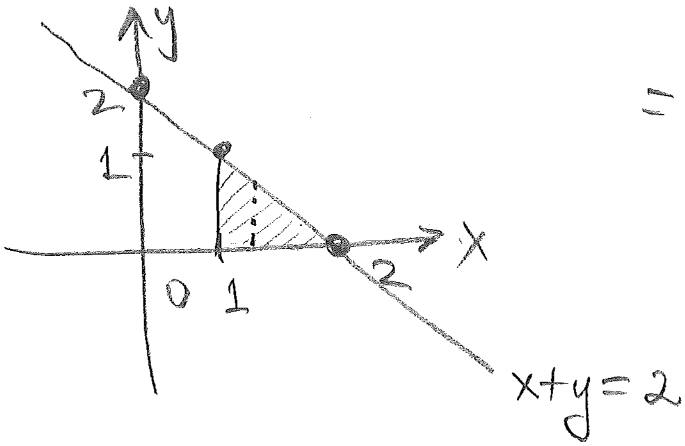
$\Rightarrow \sin\alpha = \sqrt{1-r^2}$

$\displaystyle = \int_0^1 \sqrt{1-r^2}\,dr$

$= \frac{1}{4}$ (area of circle of radius 1)

$= \frac{1}{4}\pi(1)^2 = \boxed{\frac{\pi}{4}}$

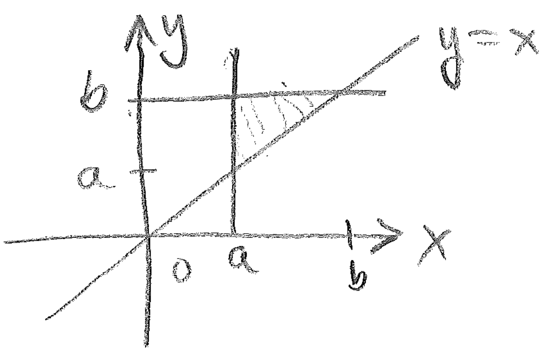
(c) $\int_0^1 \int_1^{2-y} (x+y)^2 \, dx \, dy = \int_1^2 \int_0^{2-x} (x+y)^2 \, dy \, dx$



$$= \int_1^2 \frac{(x+y)^3}{3} \Big|_{y=0}^{2-x} dx$$

$$= \int_1^2 \left(\frac{2^3}{3} - \frac{x^3}{3} \right) dx = \boxed{\frac{17}{12}}$$

(d) $\int_a^b \int_a^y f(x,y) \, dx \, dy = \int_a^b \int_x^b f(x,y) \, dy \, dx$



In terms of antiderivatives, let say $\frac{\partial F(x,y)}{\partial x} = f(x,y)$

then $I = \int_a^b \int_a^y f(x,y) \, dx \, dy = \int_a^b \int_a^y \left\{ \frac{\partial F}{\partial x}(x,y) \right\} dx \, dy$

$= \int_a^b F(x,y) \Big|_{x=a}^{x=y} dy = \int_a^b \{ F(y,y) - F(a,y) \} dy$. Now

Suppose $\frac{dH(y)}{dy} = F(y,y) - F(a,y)$ (note H is a function only of y)

then $I = \int_a^b \{ F(y,y) - F(a,y) \} dy = \int_a^b \frac{dH(y)}{dy} dy = \underline{H(b) - H(a)}$.

(cont'd)

Also since $I = \int_a^b \int_x^b f(x,y)\,dy\,dx$,

let $\dfrac{\partial P(x,y)}{\partial y} = f(x,y)$

then $I = \int_a^b \int_x^b \dfrac{\partial P(x,y)}{\partial y}\,dy\,dx = \int_a^b P(x,y)\Big|_{y=x}^{y=b} dx$

$= \int_a^b \{P(x,b) - P(x,x)\}\,dx$

let $\dfrac{dQ(x)}{dx} = P(x,b) - P(x,x)$

then $I = \int_a^b \dfrac{dQ(x)}{dx}\,dx = Q(b) - Q(a).$

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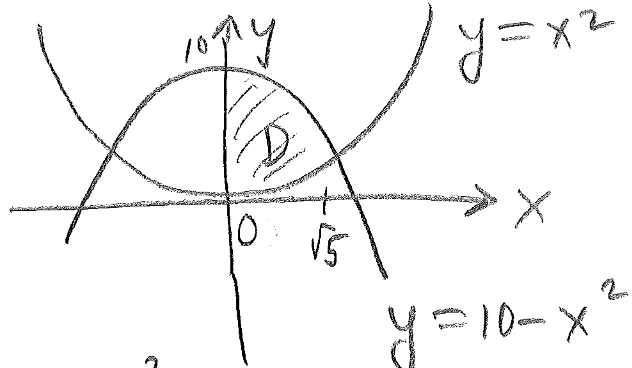


$$\therefore \iint_D y^2\sqrt{x}\,dx\,dy = \int_0^{\sqrt{5}}\int_{x^2}^{10-x^2} y^2\sqrt{x}\,dy\,dx$$

$$= \frac{1}{3}\int_0^{\sqrt{5}} \{(10-x^2)^3\sqrt{x} - x^6\sqrt{x}\}\,dx$$

let $u^2 = x \Rightarrow 2u\,du = dx$

$$\therefore \iint_D y^2\sqrt{x}\,dx\,dy = \frac{1}{3}\int_0^{5^{\frac{1}{4}}} \{(10-u^4)^3 u - u^{13}\}\,(2u\,du)$$

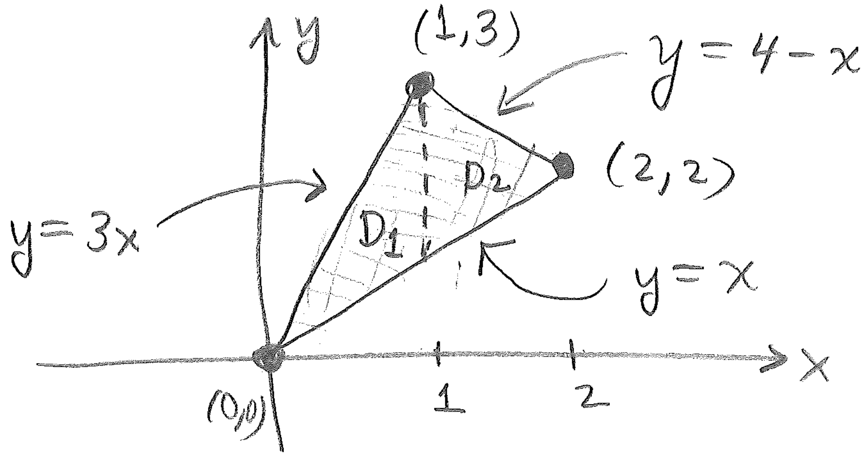
$$= \frac{2}{3}\int_0^{5^{\frac{1}{4}}} \{(10-u^4)^3 u^2 - u^{14}\}\,du$$

$$= \frac{2}{3}\int_0^{5^{\frac{1}{4}}} \{1000u^2 - 300u^6 + 30u^{10} - 2u^{14}\}\,du$$

$$= \boxed{\frac{2}{3}\left(\frac{1000}{3}5^{3/4} - \frac{300}{7}5^{7/4} + \frac{30}{11}5^{11/4} - \frac{2}{15}5^{15/4}\right)}$$

or $\dfrac{78800}{69}\,5^{34}$

Page 353 #10



$$\iint_D e^{x-y}\,dA = \iint_{D_1} e^{x-y}\,dA + \iint_{D_2} e^{x-y}\,dA$$

$$\iint_{D_1} e^{x-y}\,dA = \int_0^1\int_x^{3x} e^{x-y}\,dy\,dx = \int_0^1 -e^{x-y}\Big|_{y=x}^{3x}\,dx$$

$$= \int_0^1 \left(1-e^{-2x}\right)dx = \frac{1}{2}\left(1+e^{-2}\right)$$

$$\iint_{D_2} e^{x-y}\,dA = \int_1^2\int_x^{4-x} e^{x-y}\,dy\,dx = \int_1^2 -e^{x-y}\Big|_{y=x}^{4-x}\,dx$$

$$= \int_1^2 \left(1-e^{2x-4}\right)dx = \frac{1}{2}\left(1+e^{-2}\right)$$

$$\therefore \iint_D e^{x-y}\,dA = \boxed{1+e^{-2}}$$

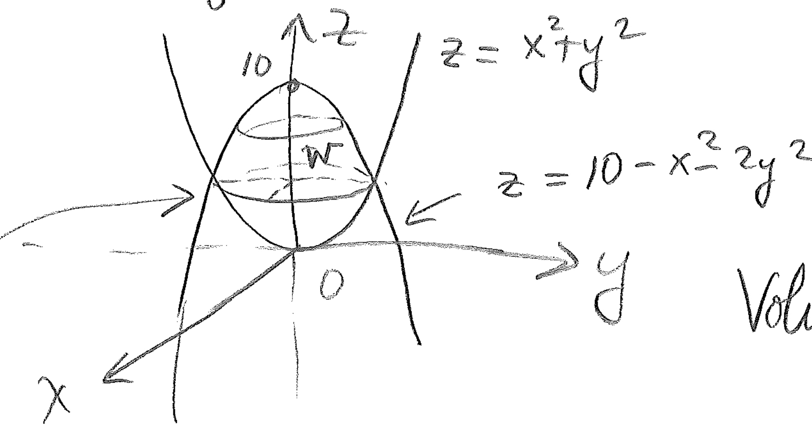
$\boxed{4}$ Page 363

#1 $\displaystyle\int_0^1\int_0^1\int_0^1 x^2\,dx\,dy\,dz = \int_0^1\int_0^1 \frac{1}{3}\,dy\,dz = \boxed{\frac{1}{3}}$

#2 $\displaystyle\int_0^1\int_0^1\int_0^1 y e^{-xy}\,dx\,dy\,dz = \int_0^1\int_0^1 -e^{-xy}\Big|_{x=0}^{1}\,dy\,dz$

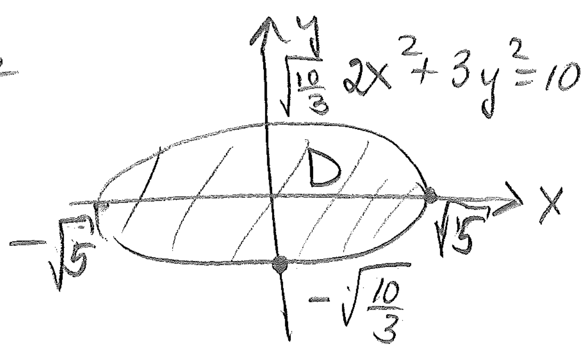
$\displaystyle= \int_0^1\int_0^1 (1-e^{-y})\,dy\,dz = \int_0^1 (y+e^{-y})\Big|_{y=0}^{1}\,dz = \boxed{\frac{1}{e}}$

#9



$\displaystyle \text{Volume} = \iiint_W dV$

$W: \begin{cases} x^2+y^2 \le z \le 10 - x^2 - 2y^2 \\ \text{where } (x,y)\in D \end{cases}$



Boundary of D is the ellipse of intersection

$x^2+y^2 = 10 - x^2 - 2y^2$

$2x^2+3y^2 = 10$

(cont'd)

$$W: \begin{cases} x^2+y^2 \le z \le 10-x^2-2y^2 \\ -\frac{1}{\sqrt{3}}\sqrt{10-2x^2} \le y \le \frac{1}{\sqrt{3}}\sqrt{10-2x^2} \\ -\sqrt{5} \le x \le \sqrt{5} \end{cases}$$

$$\therefore V = \iiint_W dV = \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\frac{1}{\sqrt{3}}\sqrt{10-2x^2}}^{\frac{1}{\sqrt{3}}\sqrt{10-2x^2}} \int_{x^2+y^2}^{10-x^2-2y^2} dz\, dy\, dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\frac{1}{\sqrt{3}}\sqrt{10-2x^2}}^{\frac{1}{\sqrt{3}}\sqrt{10-2x^2}} (10-2x^2-3y^2)\, dy\, dx$$

$$= \int_{-\sqrt{5}}^{5} 2\left\{ \frac{10}{\sqrt{3}}\sqrt{10-2x^2} - \frac{2x^2}{\sqrt{3}}\sqrt{10-2x^2} - \frac{1}{3\sqrt{3}}(10-2x^2)^{3/2} \right\} dy$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} 2\left\{ \frac{1}{\sqrt{3}}(10-2x^2)^{3/2} - \frac{1}{3\sqrt{3}}(10-2x^2)^{3/2} \right\} dy$$

$$= \frac{4}{3\sqrt{3}} \int_{-\sqrt{5}}^{\sqrt{5}} (10-2x^2)^{3/2} dx = \frac{8}{3\sqrt{3}} \int_0^{\sqrt{5}} (10-2x^2)^{3/2} dx$$

let $x = \sqrt{5}\sin\theta \Rightarrow dx = \sqrt{5}\cos\theta\, d\theta$

$$\therefore V = \frac{8}{3\sqrt{3}} \int_0^{\sqrt{5}} (10 - 2x^2)^{3/2}\, dx$$

$$= \frac{8}{3\sqrt{3}} \int_0^{\frac{\pi}{2}} (10 - 10\sin^2\theta)^{3/2} \cdot \sqrt{5}\cos d\theta$$

$$= \frac{8\sqrt{5}}{3\sqrt{3}} (10^{3/2}) \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

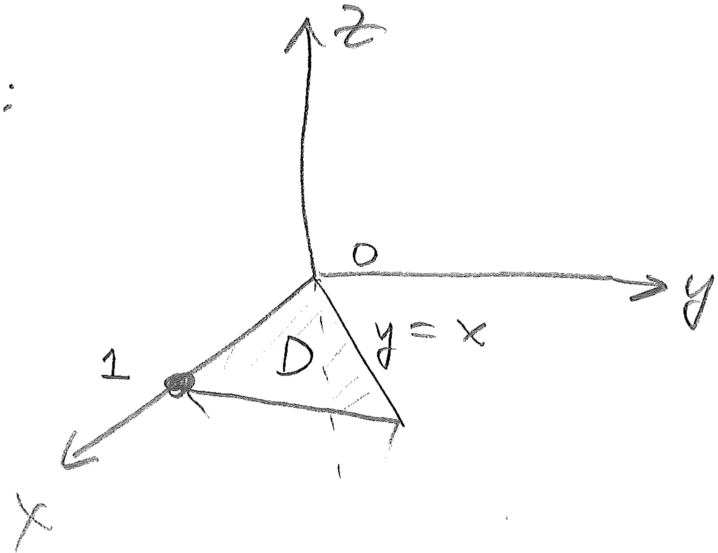
$$= \frac{8}{3}\sqrt{\frac{5}{3}(1000)} \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{4(50)}{3}\sqrt{\frac{2}{3}} \int_0^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \underbrace{\cos^2 2\theta}_{= \frac{1+\cos 4\theta}{2}}\right) d\theta$$

$$= \frac{100}{\cancel{3}}\sqrt{\frac{2}{3}}\left(\frac{\cancel{3}}{2}\right)\left(\frac{\pi}{2}\right)$$

$$= \boxed{25\pi\sqrt{\frac{2}{3}}}$$

page 363 #11:



$$W: \begin{cases} -x-y \leq z \leq 0 \\ 0 \leq y \leq x \\ 0 \leq x \leq 1 \end{cases}$$

$$\therefore\ V = \iiint_W dx\,dy\,dz = \int_0^1 \int_0^x \int_{-x-y}^{0} dz\,dy\,dx = \boxed{\frac{1}{2}}$$

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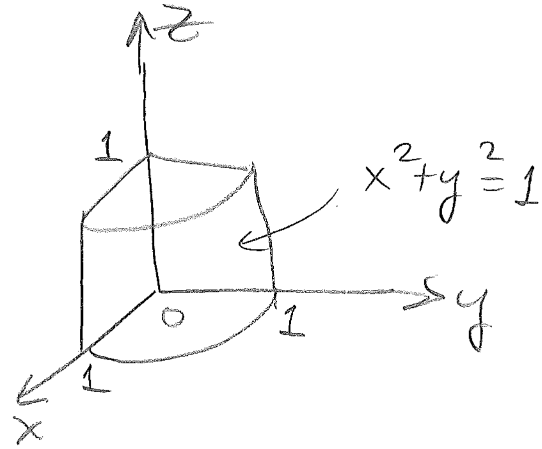


$$W: \begin{cases} 0 \le y \le \sqrt{1-x^2} \\ 0 \le x \le 1 \\ 0 \le z \le 1 \end{cases}$$

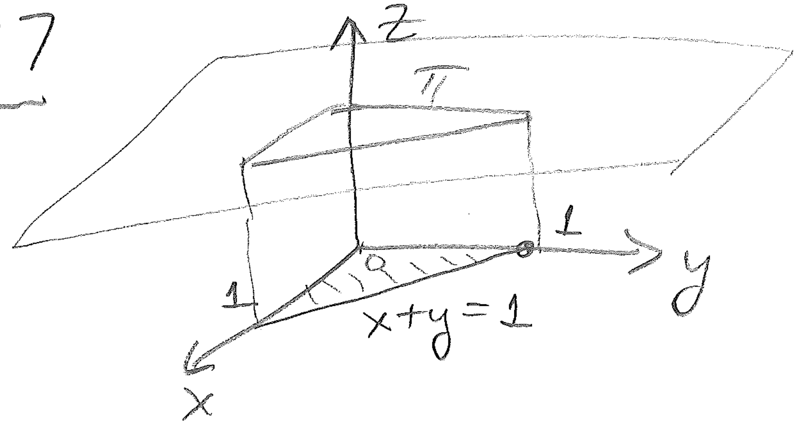
$$\therefore \iiint_W z \, dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{1-x^2}} z \; dy\, dx\, dz$$

$$= \int_0^1 \int_0^1 z\sqrt{1-x^2}\, dx\, dz = \int_0^1 \frac{1}{2}\sqrt{1-x^2}\, dx$$

$$= \frac{1}{2}\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\frac{1}{4}\right) \text{area of circle of radius } 1$$

$$= \boxed{\frac{\pi}{8}}$$

page 363 #17



$$W: \begin{cases} 0 \le z \le \pi \\ 0 \le y \le 1-x \\ 0 \le x \le 1 \end{cases}$$

$$\iiint_W x^2 \cos z \, dV = \int_0^1 \int_0^{1-x} \int_0^{\pi} x^2 \cos z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} x^2 \sin z \Big|_{z=0}^{\pi} dy \, dx = \boxed{0}$$

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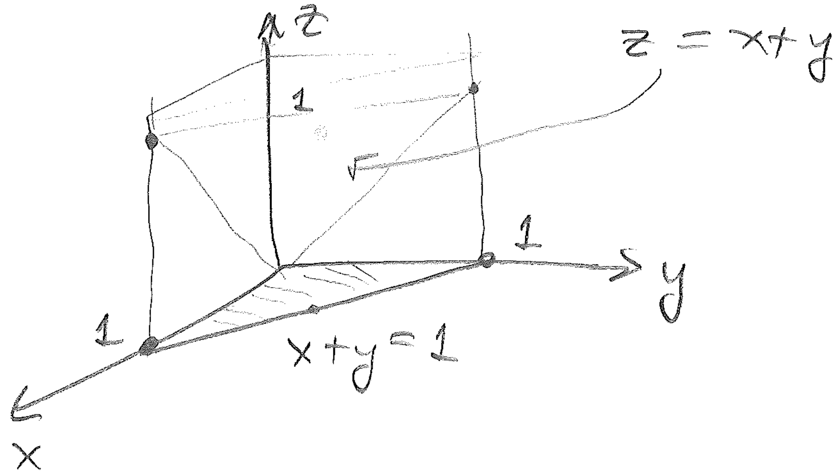
$$W: \begin{cases} 0 \le z \le x+y \\ 0 \le y \le 1-x \\ 0 \le x \le 1 \end{cases}$$



(a) $\iiint_W dV = \int_0^1 \int_0^{1-x} \int_0^{x+y} dz\,dy\,dx = \boxed{\frac{1}{3}}$

(b) $\iiint_W x\,dV = \int_0^1 \int_0^{1-x} \int_0^{x+y} x\,dz\,dy\,dx = \boxed{\frac{1}{8}}$

(c) $\iiint_W y\,dV = \int_0^1 \int_0^{1-x} \int_0^{x+y} y\,dz\,dy\,dx = \boxed{\frac{1}{8}}$