**Solutions**

MA 510 - Spring 2010

# Problem Set # 9

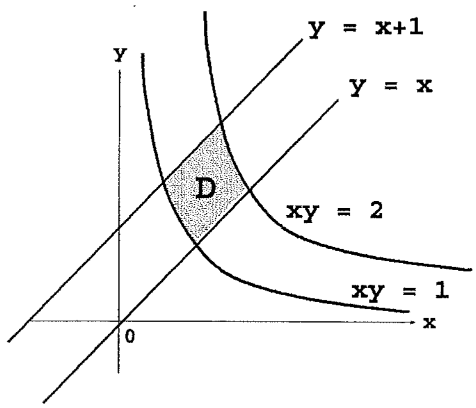
(due: April 2)

**1.** $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0}\int_{\sqrt{x^2+y^2}}^{3}(xy+2z)\,dz\,dy\,dx \;=\; \int_{\boxed{\phantom{0}}}^{\boxed{\phantom{0}}}\int_{\boxed{\phantom{0}}}^{\boxed{\phantom{0}}}\int_{\boxed{\phantom{0}}}^{\boxed{\phantom{0}}}(xy+2z)\;dy\,dx\,dz$ .

**2.** **Page 375** : # 3.

**3.** **Page 390** : # 1, 3(b), 6, 26(a), 29.

**4.** Compute $I = \displaystyle\iint_D \frac{(y+x)}{xy}\,dx\,dy$, where $D$ is the region:



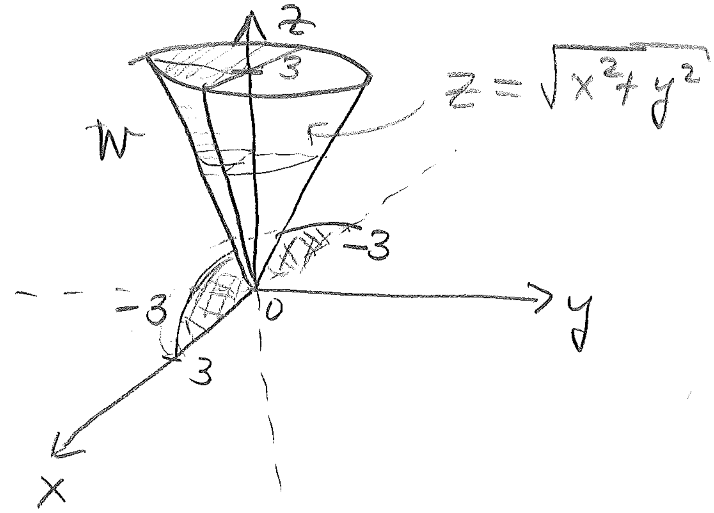

*Hint* : Let $u = xy$ and $v = y - x$ and recall that $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$

**5.** Let $W$ be that part of solid sphere $x^2+y^2+z^2 \le 9$ which lies above the plane $z = 2$. Let $I = \displaystyle\iiint_W (2x^2+2y^2+10z)\,dV$. Set up but do not evaluate the triple integral $I$ in *Rectangular, Cylindrical and Spherical Coordinates.*

# Problem Set #9

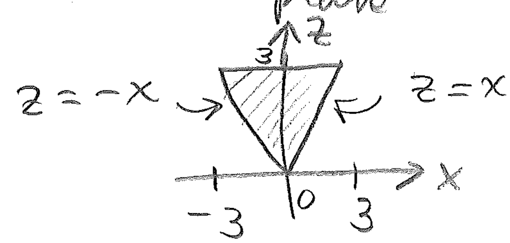
(1) $$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0}\int_{\sqrt{x^2+y^2}}^{3}(xy+2z)\,dz\,dy\,dx = I$$

$$\Rightarrow W:\begin{cases}\sqrt{x^2+y^2}\le z\le 3\\ -\sqrt{9-x^2}\le y\le 0\\ -3\le x\le 3\end{cases}$$




$$\Rightarrow W:\begin{cases}-\sqrt{z^2-x^2}\le y\le 0\\ -z\le x\le z\\ 0\le z\le 3\end{cases}$$

projection of W onto xz plane




$$\therefore \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0}\int_{\sqrt{x^2+y^2}}^{3}(xy+2z)\,dz\,dy\,dx=\int_{\boxed{0}}^{\boxed{3}}\int_{\boxed{-z}}^{\boxed{z}}\int_{\boxed{-\sqrt{z^2-x^2}}}^{\boxed{0}}(xy+2z)\,dy\,dx\,dz$$

12

Page 375 # 3 : $D^* = [0,1]\times[0,1]$

$T(u,v) = (-u^2+4u, v)$

Is $T$ one-to-one? Suppose $T(u_1,v_1) = T(u_2,v_2)$

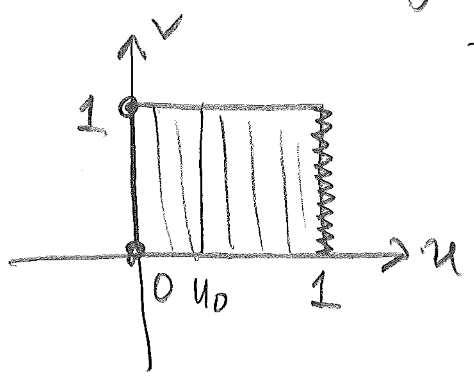
$\Rightarrow (-u_1^2+4u_1, v_1) = (-u_2^2+4u_2, v_2) \Rightarrow \begin{cases} -u_1^2+4u_1 = -u_2^2+4u_2 & (1) \\ v_1 = v_2 & (2) \end{cases}$

(1) $\Rightarrow$ $u_2^2 - u_1^2 + 4u_1 - 4u_2 = 0 \Rightarrow (u_2-u_1)(u_2+u_1-4) = 0$

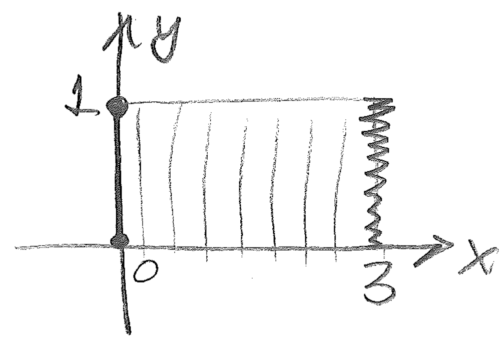
since $0 \le u_1, u_2 \le 1 \Rightarrow u_2 + u_1 - 4 \neq 0$

$\therefore u_2 = u_1$ Hence $T$ is indeed 1-1.

Find the image of $D^*$ under $T$:




$T: \begin{cases} x = -u^2+4u \\ y = v \end{cases}$



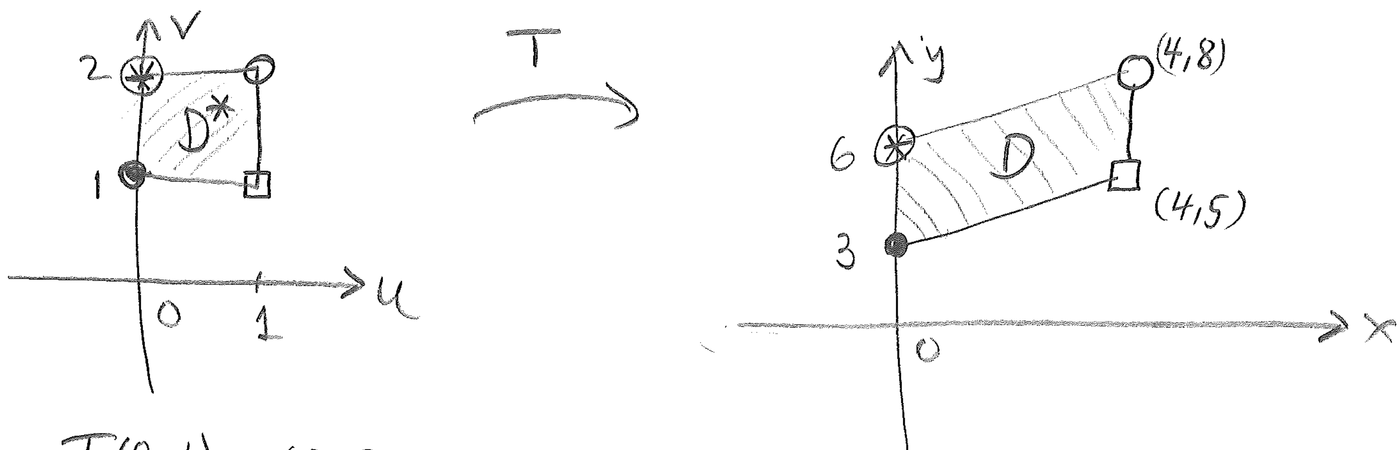

Consider image of vertical lines $u=u_0$

$\Rightarrow \begin{cases} x = -u_0^2+4u_0 \\ y = v \end{cases}$ these are vertical lines and since $0 \le v \le 1 \Rightarrow 0 \le y \le 1$

$\therefore T(D^*) = [0,3]\times[0,1]$

Page 390 #1: $\iint_D e^{(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta = \boxed{\pi(e-1)}$

Page 390 # 3(b): $T(u,v) = (4u, 2u+3v)$ is linear so it maps parallelograms to parallelograms and vertices to vertices. Hence we only need to consider the image of the vertices of $D^*$ in order to determine $D$:



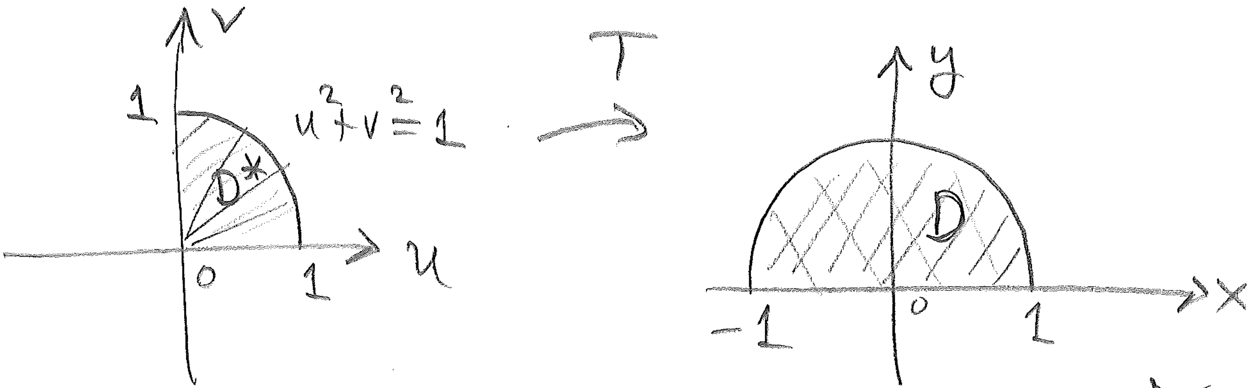

$T(0,1) = (0,3)$
$T(0,2) = (0,6)$
$T(1,1) = (4,5)$
$T(1,2) = (4,8)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4 & 0 \\ 2 & 3 \end{vmatrix} = 12$$

$$\therefore \iint_D (x-y)\,dx\,dy = \iint_{D^*} (4u-2u-3v)\,|12|\,du\,dv$$

$$= \int_1^2 \int_0^1 12(2u-3v)\,du\,dv = \boxed{-42}$$

Page 390 #6: $T(u,v) = (u^2 - v^2, 2uv)$




$$T(u,v) = (u^2 - v^2, 2uv) \Rightarrow \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

We can express points in $D^*$ using Polar coordinates so

let $u = r\cos\theta$, $v = r\sin\theta$ so $D^*: \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$

Hence $\begin{cases} x = u^2 - v^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta \\ y = 2uv = 2r^2\sin\theta\cos\theta = r^2\sin 2\theta \end{cases}$

Thus radial lines of length $r$, angle $\theta$ get mapped to radial lines of length $r^2$, angle $2\theta$. Hence $D$ is as above

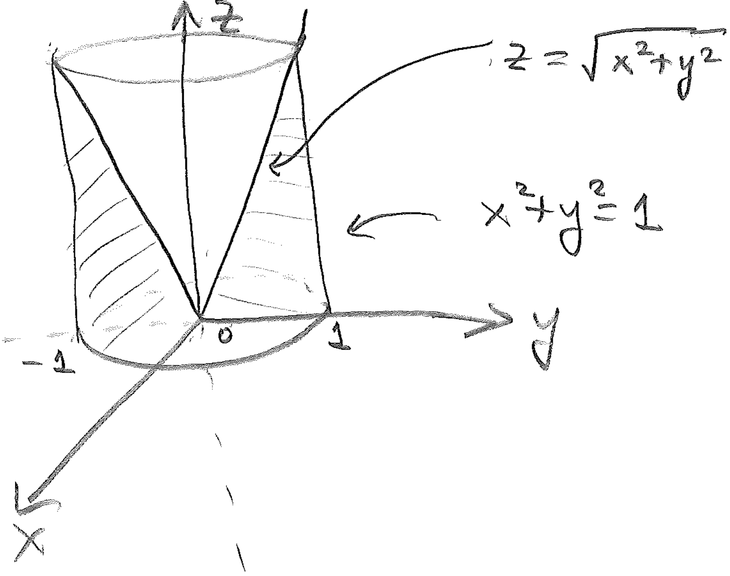
$$\iint_D dx\,dy = \boxed{\frac{\pi}{2}} \quad (\text{i.e } \tfrac{1}{2} \text{ area of circle of radius } 1)$$

Or, without knowing what $D$ is:

$$\iint_D dx\,dy = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{D^*} 4(u^2+v^2)\,du\,dv = \int_0^{\frac{\pi}{2}}\int_0^1 4r^2\, r\,dr\,d\theta$$

$$= \boxed{\frac{\pi}{2}}$$

page 390 # 26(a)




5

$$B:\begin{cases} 0 \le z \le \sqrt{x^2+y^2} \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ -1 \le x \le 1 \end{cases}$$

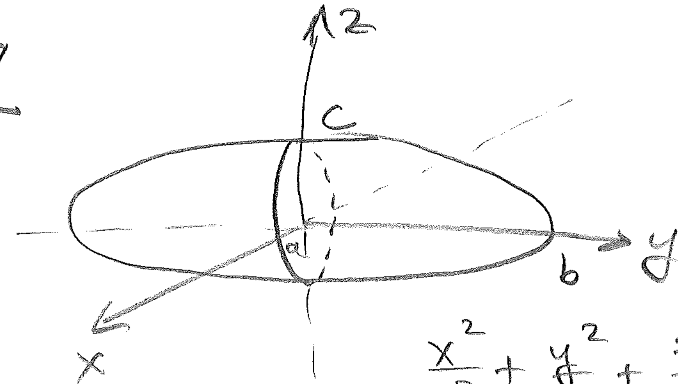
In Cylindrical Coordinates $B:\begin{cases} 0 \le z \le r \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$

$$\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = r$$

$$\therefore \iiint_B z\,dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_0^r z\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{r^3}{2}\,dr\,d\theta = \boxed{\frac{\pi}{4}}$$

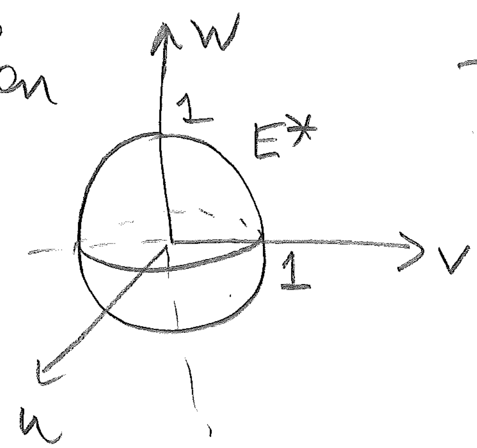
Page 390 # 29



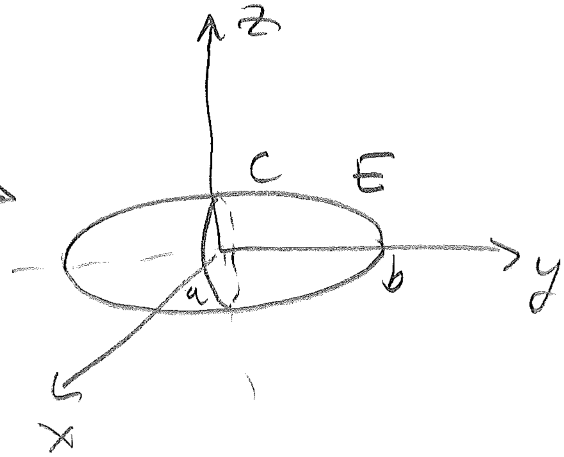

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Let $T(u,v,w)=(au, bv, cw)$

Then




$$T:\begin{cases} x=au \\ y=bv \\ z=cw \end{cases}$$




$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}= abc$$

$$\therefore \quad \text{(a)} \ \mathrm{Vol}(E)=\iiint_E dx\,dy\,dz=\iiint_{E^*}\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$=\iiint_{E^*}(abc)\,du\,dv\,dw=(abc)\underbrace{\iiint_{E^*} du\,dv\,dw}_{\text{volume of sphere of radius } 1}$$

$$=\boxed{\frac{4}{3}abc}$$

Page 390 # 29 (b)

$$I = \iiint_E \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx\,dy\,dz = \,?$$

Use $T(u,v,w) = (au, bv, cw)$ as in part (a)

So $\begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$ and $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = abc$

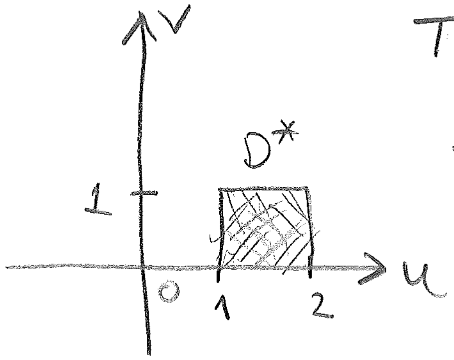
$$\therefore I = \iiint_E \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 dx\,dy\,dz = \iiint_{E^*} (u^2 + v^2 + w^2)\, abc\; du\,dv\,dw$$

Now use spherical coordinates: $\begin{cases} u = \rho\cos\theta\sin\phi \\ v = \rho\sin\theta\sin\phi \\ w = \rho\cos\phi \end{cases}$

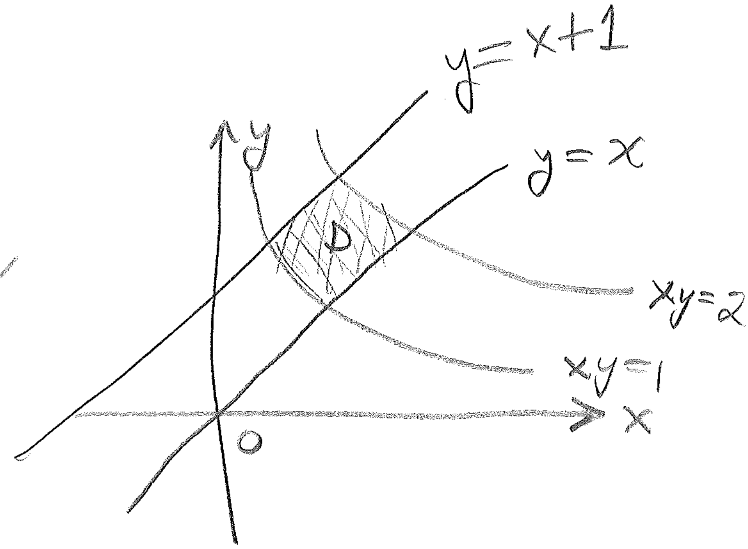
$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 (abc)\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \boxed{\frac{4\pi}{5} abc}$$

**4** $I = \iint_D \left(\frac{y+x}{xy}\right) dx\,dy$




$T^{-1}: \begin{cases} u = xy \\ v = y - x \end{cases}$




Under $T^{-1}$, since $\begin{cases} u = xy \\ v = y-x \end{cases} \Rightarrow$ $y=x$ maps to $v=0$

$y = x+1$ maps to $v = 1$

$xy = 1$ maps to $u = 1$

$xy = 2$ maps to $u = 2$

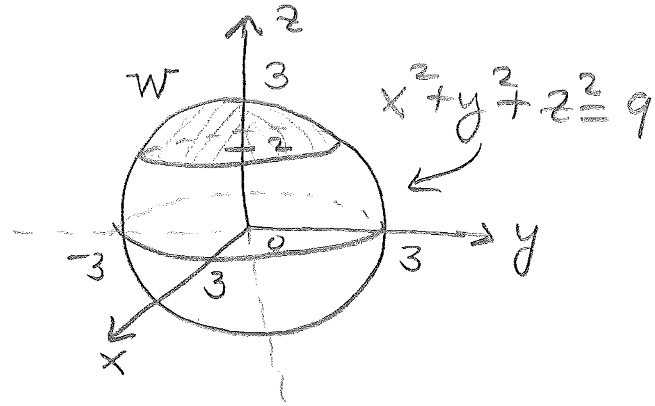
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -1 & 1 \end{vmatrix} = y + x \quad \therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{y+x}$$

Thus, $$\iint_D \left(\frac{y+x}{xy}\right) dx\,dy = \iint_{D^*} \left(\frac{y(u,v) + x(u,v)}{x(u)\,y(u)}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_{D^*} \left(\frac{y+x}{u}\right) \frac{1}{(y+x)}\, du\,dv = \iint_{D^*} \frac{1}{u}\, du\,dv$$

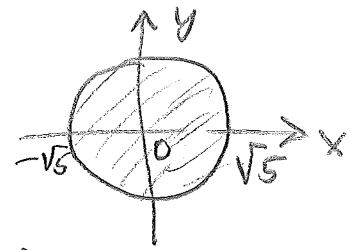
$$= \int_0^1 \int_1^2 \frac{1}{u}\, du\,dv = \boxed{\ln 2}$$

$\boxed{5}$ $I = \iiint\limits_W (2x^2 + 2y^2 + 10z)\, dV$




$\boxed{RC}$ $W: \begin{cases} 2 \le z \le \sqrt{9-x^2-y^2} \\ -\sqrt{5-x^2} \le y \le \sqrt{5-x^2} \\ -\sqrt{5} \le x \le \sqrt{5} \end{cases}$

projection in xy plane




$x^2+y^2+z^2=9$

$z=2$

$\Rightarrow x^2+y^2=5$

$\therefore$ In Rectangular Coordinates

$$I = \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}} \int_{2}^{\sqrt{9-x^2-y^2}} (2x^2 + 2y^2 + 10z)\, dz\, dy\, dx \checkmark$$

(cont'd)

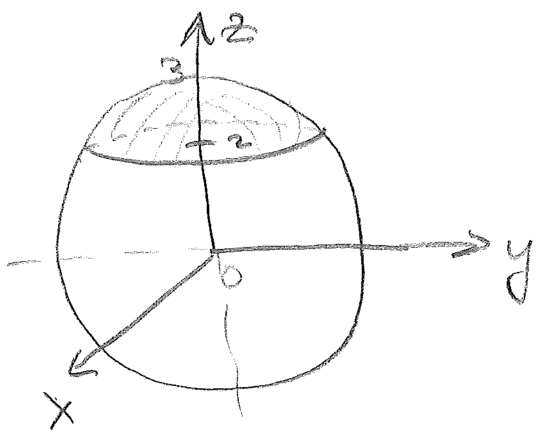
$\boxed{CC}$

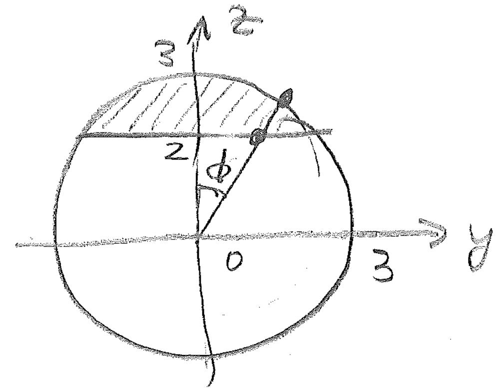
$$W: \begin{cases} 2 \le z \le \sqrt{9-r^2} \\ 0 \le r \le \sqrt{5} \\ 0 \le \theta \le 2\pi \end{cases}$$

$\therefore$ In Cylindrical Coordinates

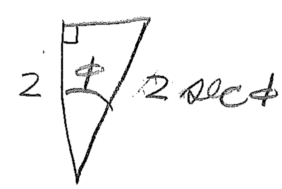
$$I = \int_0^{2\pi}\int_0^{\sqrt{5}}\int_2^{\sqrt{9-r^2}} (2r^2 + 10z)\, r\, dz\, dr\, d\theta \quad \checkmark$$

$\boxed{SC}$




In spherical coordinates




$$W: \begin{cases} 2\sec\phi \le \rho \le 3 \\ 0 \le \phi \le \cos^{-1}\left(\frac{2}{3}\right) \\ 0 \le \theta \le 2\pi \end{cases}$$

max $\phi$ occurs when

$x^2+y^2+z^2 = 9$ and $z = 2$

$\rho^2 = 9$ and $\rho\cos\phi = 2$

$\Rightarrow \rho = 3$ so $\phi = \cos^{-1}\left(\frac{2}{3}\right)$

(cont'd)

$\therefore$ In Spherical Coordinates

$$I=\iiint\limits_W (2x^2+2y^2+10z)\,dx\,dy\,dz=\int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{2}{3}\right)}\int_{2\sec\phi}^{3}\left(2\rho^2\sin^2\phi+10\rho\cos\phi\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \checkmark$$