# APPLICATIONS OF EXTREME
# POINTS TO DISTORTION ESTIMATES

**by**

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ABSTRACT. Sharp distortion theorems for close-to-convex functions which satisfy Re $f'(z) > \gamma$ are proved using extreme points.

## 1. Introduction

Let $\mathcal{H}(\Delta)$ be the class of all functions which are analytic in the unit disk $\Delta$. For $0 \le \gamma < 1$, we let $R_\gamma$ denote the subclass of functions $f \in \mathcal{H}(\Delta)$ which satisfy the condition Re $f'(z) > \gamma$ for all $z \in \Delta$, with the normalization $f(0) = 0$ and $f'(0) = 1$. By the Noshiro-Warschawski-Wolff Theorem (see [2] for example), it is well-known that all functions in $R_\gamma$ are univalent and belong to the classical family of univalent functions $S$. In fact, functions in $R_\gamma$ are close-to-convex. The class $R_0$ has been studied and many results known ([7]). Thomas [9] proved that if $f \in R_0$ and $|z| = r < 1$, then

$$\left| \frac{z f'(z)}{f(z)} \right| \le \frac{K}{-(1-r)\log(1-r)}$$

and stated that extreme point methods did not seem to give sharp bounds. Later, London [6] found the sharp bound without using extreme points. The purpose of this paper is to show that extreme point methods can be used to both generalize London's result for all $0 \le \gamma < 1$ and give sharp estimates for other distortion results. The basic method given here may be used in other types of extremal problems for other families of functions.

## 2. Main Results

For each $0 \le \gamma < 1$, for convenience only, we henceforth set

$$(2.1) \qquad\qquad \alpha = 1 - 2\gamma$$

and define functions $k_\alpha \in R_\gamma$ as follows :

$$(2.2) \qquad\qquad k_\alpha(z) = -\alpha z - (1 + \alpha)\log(1 - z).$$

These and their rotations will turn out to be extremal functions. We can now state our results.

**Theorem 2.1.** *If $f \in R_\gamma$, $0 \le \gamma < 1$ and $|z| = r < 1$, then*

$$\frac{|f'(z)|}{Re\left\{\frac{f(z)}{z}\right\}} \le \frac{r k'_\alpha(r)}{k_\alpha(r)} = \frac{r(1 + \alpha\, r)}{(1 - r)[-\alpha\, r - (1 + \alpha) \log(1 - r)]}\,.$$

*Equality holds for $0 < r < 1$ when $f(z) = k_\alpha(z)$.*

The above result also gives the sharp upper bound for both $|z f'(z)/f(z)|$ and for $Re\ \{f'(z)\}/\ Re\ \{f(z)/z\}$.

**Theorem 2.2.** *If $f \in R_\gamma$, $0 \le \gamma < 1$ and $|z| = r < 1$, then*

$$\frac{|z f''(z)|}{Re\ \{f'(z)\}} \le \frac{r k''_\alpha(r)}{k'_\alpha(r)} = \frac{r(1 + \alpha)}{(1 - r)(1 + \alpha\, r)}\,.$$

*Equality holds for $0 < r < 1$ when $f(z) = k_\alpha(z)$.*

It is known (see [1] or[5]) that if $Re\ f'(z) > \gamma$, then $Re\ \{f(z)/z\} > \gamma$. However, the sharp estimate is given by :

**Theorem 2.3.** *If $f \in R_\gamma$, $0 \le \gamma < 1$ and $|z| = r < 1$, then*

$$Re\ \left\{\frac{f(z)}{z}\right\} \ge -\frac{k_\alpha(-r)}{r} = -\alpha + \frac{(1 + \alpha)}{r} \log(1 + r)\,.$$

*Equality holds for $0 < r < 1$ when $f(z) = -k_\alpha(-z)$.*

## 3. Proofs

There are some elementary properties of the class $R_\gamma$ which we now state. It is clear that $R_\gamma$ is a compact family in $\mathcal{H}(\Delta)$ (endowed with the topology of uniform convergence on compacta in $\Delta$). It is a convex family, i.e., if $f, g \in R_\gamma$, then $tf + (1 - t)g \in R_\gamma$ for all $t \in [0, 1]$. This class is also rotationally invariant : $f \in R_\gamma$ if and only if $e^{-i\nu} f(e^{i\nu} z) \in R_\gamma$ for all $\nu \in \mathbb{R}$. Observe that the derivatives of functions in $R_\gamma$ are related to functions in the classical Carathéordory class $\mathcal{P}$ by $f \in R_\gamma$ if and only if $(f' - \gamma)/(1 - \gamma) \in \mathcal{P}$. From this and the well-known fact that the extreme points of $\mathcal{P}$ are given by

$$\mathcal{E}(\mathcal{P}) = \left\{\frac{1 + e^{i\mu} z}{1 - e^{i\mu} z} : 0 \le \mu \le 2\pi\right\},$$

it follows that the extreme points of $R_\gamma$ are

(3.1) $$\mathcal{E}(R_\gamma) = \left\{e^{-i\mu} k_\alpha(e^{i\mu} z) : 0 \le \mu \le 2\pi\right\}\,.$$

We need two preliminary results. The first may be compared here with that of Ruscheweyh [8]. It is the convexity of certain functionals which will be important to prove our results. (Recall that a functional $\Phi$ is convex if $\Phi(tf + (1 - t)g) \le t\Phi(f) + (1 - t)\Phi(g)$ for $f, g \in \mathcal{H}(\Delta)$ and all $t \in [0, 1]$.) The second contains the computations necessary to complete our proofs.

**Lemma 1.** *Let $\mathcal{F}$ be a compact and convex family of functions in $\mathcal{H}(\Delta)$. If $\Phi$ and $L$ are real-valued continuous functionals on $\mathcal{H}(\Delta)$ which are convex and linear, respectively, with $L(\mathcal{F}) > 0$ and $|\Phi(\mathcal{F})| < \infty$, then*

$$\max_{f \in \mathcal{F}} \frac{\Phi(f)}{L(f)} = \max_{f \in \mathcal{E}(\mathcal{F})} \frac{\Phi(f)}{L(f)}.$$

**Lemma 2.** *If $J(t, x) \equiv 1 + t^2 - 2tx$ and $0 < r < 1$, then*

$(i)$ $\displaystyle\int_0^r \frac{\rho\,(1 - \rho^2)}{J(\rho, x)^2}\, d\rho \leq \frac{r^2}{J(r, x)}\,, \quad -1 \leq x \leq 1$

$(ii)$ $\displaystyle\int_0^r \frac{\rho\,(1 - \rho^2)}{J(\rho, x)^2}\, d\rho \leq \left[\frac{r(1 + r^2)}{J(r, -x)\,J(r, x)}\right] \int_0^r \frac{(1 - \rho^2)}{J(\rho, x)}\, d\rho\,, \quad -1 \leq x \leq 1$

$(iii)$ $\displaystyle\int_0^r \frac{\rho\,(1 - \rho^2)}{J(\rho, x)^2}\, d\rho \leq \left[\frac{r}{J(r, x)}\right] \int_0^r \frac{(1 - \rho\,x)}{J(\rho, x)}\, d\rho\,, \quad -1 \leq x \leq 1$

$(iv)$ $\displaystyle\int_0^r \frac{(\rho\,x - \rho^2)}{J(\rho, x)}\, d\rho \leq \left[\frac{1 + \alpha\,r^2}{J(\alpha\,r, -x)}\right] \int_0^r \frac{(1 - \rho\,x) + \alpha\,(\rho\,x - \rho^2)}{J(\rho, x)}\, d\rho\,, \quad 0 \leq \alpha,\ x \leq 1.$

We proceed now with the proofs of the main results. The lemmas are proved at the end of this section.

**Proof of Theorem 2.1.** Since $R_\gamma$ is rotationally invariant, it suffices to prove the inequality for $z = r$. Thus define $\Phi(f) \equiv |f'(r)|$ and $L(f) \equiv \operatorname{Re}\{f(r)/r\}$. Note that $L(R_\gamma) > \gamma$. Applying Lemma 1, using (2.2) and (3.1), we conclude that for every $f \in R_\gamma$,

(3.2) $$\frac{|f'(r)|}{\operatorname{Re}\left\{\frac{f(r)}{r}\right\}} \leq \max_{0 \leq \theta \leq 2\pi} F(\theta),$$

where

(3.3) $$F(\theta) = \frac{\left|\frac{1 + \alpha\,r e^{i\theta}}{1 - r e^{i\theta}}\right|}{\operatorname{Re}\left\{-\alpha - (re^{i\theta})^{-1}(1 + \alpha)\log(1 - re^{i\theta})\right\}}.$$

By symmetry, it is without loss of generality to suppose that $0 \leq \theta \leq \pi$. The result is proved if we show that $F(\theta) \leq F(0)$.

If we let $J(t, x) = 1 + t^2 - 2tx$, where $x = \cos\theta$, then

$$\operatorname{Re}\left\{-\alpha - \frac{(1 + \alpha)}{re^{i\theta}}\log(1 - re^{i\theta})\right\} = \frac{1}{r} \int_0^r \frac{(1 - \rho\,x) + \alpha(\rho\,x - \rho^2)}{J(\rho, x)}\, d\rho$$

and

$$\frac{\partial}{\partial \theta} \operatorname{Re} \left\{ -\alpha - \frac{(1+\alpha)}{re^{i\theta}} \log(1 - re^{i\theta}) \right\} = -\frac{(1+\alpha)}{r}(\sin \theta) \int_0^r \frac{\rho(1-\rho^2)}{J(\rho, x)^2} \, d\rho$$

and also

$$\frac{\partial \left| \frac{1+\alpha\, re^{i\theta}}{1 - re^{i\theta}} \right|}{\partial \theta} = -\frac{r(1+\alpha)(1+\alpha\, r^2)\sin \theta}{J(\alpha\, r, -x)\, J(r, x)} \left| \frac{1 + \alpha\, re^{i\theta}}{1 - re^{i\theta}} \right| .$$

It follows that

(3.4) $$\qquad F'(\theta) = \frac{(1+\alpha)}{r(\operatorname{Re}\{k_\alpha(re^{i\theta})/(re^{i\theta})\})^2}(\sin \theta) \left| \frac{1 + \alpha\, re^{i\theta}}{1 - re^{i\theta}} \right| H(x),$$

where

$$H(x) = \left[ \int_0^r \frac{\rho(1-\rho^2)}{J(\rho, x)^2} \, d\rho \right] - \left[ \frac{r(1+\alpha\, r^2)}{J(\alpha\, r, -x)\, J(r, x)} \right] \int_0^r \frac{(1 - \rho\, x) + \alpha(\rho\, x - \rho^2)}{J(\rho, x)} \, d\rho,$$

with $x = \cos \theta$ and $0 \le \theta \le \pi$. It is now necessary to consider cases.

_Case 1 : $0 \le \alpha \le 1$ and $-1 \le x \le 1$._ From the above, it is enough to show that $H(x) \le 0$, or equivalently $\{J(\alpha\, r, -x)\, J(r, x)/r\}\, H(x) \le 0$, which is true if and only if $G(\alpha) \le 0$, where

(3.5) $$\qquad\qquad\qquad\qquad G(\alpha) = A\alpha^2 + B\alpha + C$$

and

$$A = r^2 \left[ \frac{J(r, x)}{r} I - \int_0^r \frac{\rho\, x - \rho^2}{J(\rho, x)} \, d\rho \right]$$

$$B = 2x J(r, x)\, I - \int_0^r \frac{r^2(1 - \rho\, x) + (\rho\, x - \rho^2)}{J(\rho, x)} \, d\rho$$

$$C = \frac{J(r, x)}{r} I - \int_0^r \frac{(1 - \rho\, x)}{J(\rho, x)} \, d\rho$$

with $I = \int_0^r \frac{\rho(1-\rho^2)}{J(\rho, x)^2} \, d\rho$. By Lemma 2 $(ii)$ and $(iii)$, respectively, we get $G(1) \le 0$ and $G(0) \le 0$. Hence if $A \ge 0$, we have $G(\alpha) \le 0$ and so $H(x) \le 0$. Hence suppose $A < 0$, that is

(3.6) $$\qquad\qquad\qquad\qquad \frac{J(r, x)}{r} I < \int_0^r \frac{\rho\, x - \rho^2}{J(\rho, x)} \, d\rho .$$

In order for (3.6) to hold, note that we must necessarily have $0 \le x \le 1$. Hence to show that $H(x) \le 0$, using (3.6) we need only verify if $0 \le x \le 1$ and $0 \le \alpha \le 1$, then

$$\int_0^r \frac{\rho\, x - \rho^2}{J(\rho, x)} \, d\rho \le \left( \frac{1 + \alpha\, r^2}{J(\alpha\, r, -x)} \right) \int_0^r \frac{(1 - \rho\, x) + \alpha(\rho\, x - \rho^2)}{J(\rho, x)} \, d\rho .$$

This is Lemma 2 $(iv)$. Hence $H(x) \le 0$, $F'(\theta) \le 0$ and so $F(\theta) \le F(0)$ .

*Case 2* : $-1 < \alpha \le 0$ and $\dfrac{r(1-\alpha)}{2} \le x \le 1$. In this case we must have

$$\frac{1 + \alpha\, r^2}{J(\alpha\, r, -x)} \ge 1$$

and hence $H(x) \le 0$ (and also $F'(\theta) \le 0$) follows if we show that

$$\int_0^r \frac{\rho(1-\rho^2)}{J(\rho, x)^2}\, d\rho \le \frac{r}{J(r,x)} \int_0^r \frac{(1-\rho\, x) + \alpha(\rho\, x - \rho^2)}{J(\rho, x)}\, d\rho \equiv M(\alpha)\,.$$

Since $M$ is linear in $\alpha$, the above inequality holds by Lemma 2 $(i)$ and $(iii)$.

*Case 3* : $-1 < \alpha \le 0$ and $-1 \le x \le \dfrac{r(1-\alpha)}{2}$. In this situation, we see that

$$\left| \frac{1 + \alpha\, re^{i\theta}}{1 - re^{i\theta}} \right| \le 1$$

and so the expression

$$\left| \frac{1 + \alpha\, re^{i\theta}}{1 - re^{i\theta}} \right| \left\{ -\alpha - \frac{(1+\alpha)}{r} \log(1-r) \right\} - \left( \frac{1 + \alpha\, r}{1 - r} \right)\ \mathrm{Re}\ \left\{ -\alpha - \frac{(1+\alpha)}{re^{i\theta}} \log(1 - re^{i\theta}) \right\}$$

is bounded above by $H(\alpha, \theta)$, where

$$H(\alpha, \theta) = \left\{ -\alpha - \frac{(1+\alpha)}{r} \log(1-r) \right\} - \left( \frac{1 + \alpha\, r}{1 - r} \right)\ \mathrm{Re}\ \left\{ -\alpha - \frac{(1+\alpha)}{re^{i\theta}} \log(1 - re^{i\theta}) \right\}\,.$$

We can show directly that $H(\alpha, \theta) \le 0$, which is equivalent to $F(\theta) \le F(0)$. For convenience, set $\mu(\theta) = \mathrm{Re}\ \{(re^{i\theta})^{-1} \log(1 - re^{i\theta})\}$. Because $\frac{1}{z} \log(1 - z)$ is a convex univalent function which maps disks $|z| \le r$ onto convex regions symmetric with respect to the real axis, we get

$$(3.7) \qquad\qquad \mathrm{Re}\ \{(re^{i\theta})^{-1} \log(1 - re^{i\theta})\} \le -\frac{1}{r} \log(1 + r)\,.$$

Now we have

$$\frac{\partial H}{\partial \alpha} = \frac{r(1 + 2\alpha)}{1 - r} - \frac{1}{r} \log(1 - r) + \mu \left[ \frac{1 + (1 + 2\alpha)r}{1 - r} \right]\,.$$

This function is linear in $\mu$ and (3.7) gives $\mu(\theta) \le -\frac{1}{r} \log(1 + r)$ and hence

$$r(1 - r) \frac{\partial H}{\partial \alpha} \le r^2(1 + 2\alpha) - (1 - r) \log(1 - r) - [1 + (1 + 2\alpha)r] \log(1 + r)\,.$$

The right hand side is linear in $\alpha$ so a brief calculation then gives $\frac{\partial H}{\partial \alpha} \le 0$. We are done because $H(\alpha, \theta) < H(-1, \theta) = 0$.

**Proof of Theorem 2.2** . As in the proof of the previous theorem, it suffices to prove our result for $z = r$. Put $\Phi(f) \equiv |r\, f''(r)|$, $L(f) \equiv \mathrm{Re}\ \{f'(r)\}$ and apply Lemma 1 with (3.1) to conclude that for any $f \in R_\gamma$,

$$\frac{|r\,f''(r)|}{\mathrm{Re}\,\{f'(r)\}} \le \max_{0 \le \theta \le 2\pi} K(\theta)\,,$$

where

(3.8) $$K(\theta) = \frac{r(1+\alpha)}{|1 - re^{i\theta}|^2} \Big/ \mathrm{Re}\,\left\{\frac{1 + \alpha\,re^{i\theta}}{1 - re^{i\theta}}\right\}\,.$$

We wish to show that $K(\theta) \le K(0)$, but this is equivalent to

$$(1 - r)(1 + \alpha\,r) - |1 - re^{i\theta}|^2\,\mathrm{Re}\,\left\{\frac{1 + \alpha\,re^{i\theta}}{1 - re^{i\theta}}\right\} \le 0.$$

This in turn is equivalent to $(1 - \alpha)(\cos\theta - 1) \le 0$. The proof of the theorem is complete.

**Proof of Theorem 2.3 .** We prove this result for $z = r$. Here we let $\Phi(f) \equiv 1$ and $L(f) \equiv \mathrm{Re}\,\{f(r)/r\}$. From Lemma 1 and (3.1) we conclude that for any $f \in R_\gamma$,

(3.9) $$\mathrm{Re}\,\left\{\frac{f(r)}{r}\right\} \ge \min_{0 \le \theta \le 2\pi} N(\theta)\,,$$

where $N(\theta) \equiv \mathrm{Re}\,\{k_\alpha(re^{i\theta})/(re^{i\theta})\}$. We need only prove that $N(\theta) \ge N(\pi)$, which is equivalent to $\mathrm{Re}\,\{(re^{i\theta})^{-1}\log(1 - re^{i\theta})\} \le -\frac{1}{r}\log(1 + r)$. This is (3.7).

We now turn our attention to the proofs of the lemmas. A very useful result due to Hallenbeck and MacGregor[4,p45] is as follows:

**Proposition.** *Let $\mathcal{F}$ be a compact family in $\mathcal{H}(\Delta)$. If $J$ is a real-valued continuous convex functional on $CH(\mathcal{F})$, the closed convex hull of $\mathcal{F}$, then*

$$\max_{f \in \mathcal{F}} J(f) = \max_{f \in \mathcal{E}(CH(\mathcal{F}))} J(f)\,.$$

**Proof of Lemma 1 .** Since $\mathcal{F}$ is compact, there always exists an extremal function $g \in \mathcal{F}$ such that

$$M = \sup_{f \in \mathcal{F}} \frac{\Phi(f)}{L(f)} = \frac{\Phi(g)}{L(g)}\,.$$

(By hypothesis $M$ is finite.) If we let $J(f) \equiv \Phi(f) - M\,L(f)$, then $J$ is a continuous real-valued convex functional whose maximum on $\mathcal{F}$ is zero. In our case $\mathcal{F}$ is a convex family hence by the above Proposition there exists $g^* \in \mathcal{E}(\mathcal{F})$ such that $J(g^*) = 0$, i.e., for any $f \in \mathcal{F}$,

$$\frac{\Phi(f)}{L(f)} \le M = \frac{\Phi(g^*)}{L(g^*)}\,.$$

The proof of this lemma is complete.

**Proof of Lemma 2 .** We consider each inequality separately.

$(i)$: This inequality holds if $\int_0^r \frac{\partial}{\partial t}\{G(t,x)\}\,dt \le 0$, where

$$G(t,x) = \int_0^t \frac{\rho\,(1-\rho^2)}{J(\rho,x)^2}\,d\rho - \frac{t^2}{J(t,x)}\,.$$

But clearly we see that $\dfrac{\partial G}{\partial t} = -t/J(t,x) \le 0,$ so $(i)$ is proved.

$(ii)$: This inequality is proved in London [6,p523].

$(iii)$: This inequality is true if $\int_0^r \frac{\partial}{\partial t}\{G(t,x)\}\,dt \le 0,$ where

$$G(t,x) = \int_0^t \frac{\rho(1-\rho^2)}{J(\rho,x)^2}\,d\rho - \frac{t}{J(t,x)} \int_0^t \frac{1-\rho\,x}{J(\rho,x)}\,d\rho\,.$$

It follows that

$$J^2(t,x)\frac{\partial G}{\partial t} = -(1-t^2)\int_0^t \frac{1-\rho\,x}{J(\rho,x)}\,d\rho + t^2(x-t)\,.$$

If $-1 \le x \le 0,$ then $\frac{\partial G}{\partial t} \le 0$ and so $(iii)$ holds in this case.

Suppose now that we have $0 \le x \le 1$. By $(ii)$, to prove $(iii)$ we need only verify that

$(3.10)$
$$\frac{1+r^2}{J(r,-x)} \int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho \le \int_0^r \frac{1-\rho\,x}{J(\rho,x)}\,d\rho\,,$$

which is true if and only if

$(3.11)$
$$\int_0^r \frac{\Psi(x,\rho)}{J(\rho,x)}\,d\rho \le 0\,,$$

where

$$\Psi(x,\rho) = (1+r^2)(1-\rho^2) + (\rho\,x - 1)J(r,-x) = Ax^2 + Bx + C\,.$$

Now clearly $A = 2\rho\,r \ge 0$, $\Psi(0,\rho) = -\rho^2(1+r^2)$, and, since $0 \le \rho \le r$, we get $\Psi(1,\rho) = (1-\rho)(\rho + r^2\rho - 2r) \le (1-\rho)(r^3 - r)$. Hence $\Psi(x,\rho) \le 0$ for $0 \le x \le 1$, so $(3.11)$ and $(3.10)$ hold and thus $(iii)$ holds.

$(iv)$: After multiplying both sides by $J(\alpha\,r, -x)$, we see that this inequality holds if and only if $K(\alpha,x) \le 0$, where

$$K(\alpha,x) = \int_0^r \frac{\alpha\left[(2rx-1)(\rho\,x-\rho^2) - r^2(1-\rho\,x)\right] - J(\rho,x)}{J(\rho,x)}\,d\rho\,.$$

Clearly $K(0,x) \le 0$ and $K(1,x) = \int_0^r \frac{\omega(x)}{J(\rho,x)}\,d\rho$, where

$$\omega(x) = (2rx-1)(\rho\,x - \rho^2) - r^2(1-\rho\,x) - J(\rho,x) = Ax^2 + Bx + C\,,$$

with $A = 2\rho\,r \ge 0$, $\omega(0) = -(1+r^2)$ and $\omega(1) \le (1-\rho)(r^2-1)$. Hence we get $K(1,x) \le 0$ and since $K$ is linear in $\alpha$, this proves $K(\alpha,x) \le 0$ and $(iv)$ is proved.

## 4. Remarks

(1) By tracing through the cases of equality, we have shown that equality in all our main results are in fact achieved *only* for the indicated extremal functions.

(2) It is essential that we separate cases in the proof of Theorem 2.1 since $F'(\theta) \leq 0$ is false for all $0 \leq \theta \leq \pi$ when $-1 < \alpha < 0$. For example, if $\alpha = -0.8$, $r = 0.8$ and $\theta_0 = 1.4266$ then $F'(\theta_0) > 0$. Also, we note that cases $(ii)$ and $(iii)$ are completely independent of each other.

(3) Theorem 2.1 is a rather curious result since both $|f'(z)|$ and $\mathrm{Re}\,\{f(z)/z\}$ are maximized for $k_\alpha(z)$ (and both minimized for $-k_\alpha(-z)$), while their quotient $|f'(z)|/\,\mathrm{Re}\,\{f(z)/z\}$ is maximized by the single function $k_\alpha(z)$ for each $0 \leq \gamma < 1$. This is not always true for estimates of other such functionals over $R_\gamma$. Indeed, it is of interest to observe that for fixed $z \in \Delta$,

$$\max_{f \in R_\gamma} \frac{\mathrm{Re}\,\{f(z)/z\}}{\mathrm{Re}\,\{f'(z)\}}$$

is not always attained for the single function $-k_\alpha(-z)$ (or $k_\alpha(z)$) for all $0 \leq \gamma < 1$. Upon selecting $z = 0.85$, $\alpha = 0$ (i.e., $\gamma = \frac{1}{2}$) and $\mu = \frac{\pi}{2}$, a numerical check shows that

$$\frac{\mathrm{Re}\,\{k_\alpha(r)/r\}}{\mathrm{Re}\,\{k'_\alpha(r)\}} < \frac{\mathrm{Re}\,\{k_\alpha(-r)/(-r)\}}{\mathrm{Re}\,\{k'_\alpha(-r)\}} < \frac{\mathrm{Re}\,\{k_\alpha(re^{i\mu})/(re^{i\mu})\}}{\mathrm{Re}\,\{k'_\alpha(re^{i\mu})\}}\,.$$

Thus by Theorem 2.1, the maximum of the the functional $\mathrm{Re}\,\{f'(z)\}/\,\mathrm{Re}\,\{f(z)/z\}$ is attained for $f(z) = k_\alpha(z)$ while the minimum is not attained (for all $0 \leq \gamma < 1$) for $f(z) = -k_\alpha(-z)$ (or $k_\alpha(z)$) as one might expect.

(4) Theorem 2.3 was proved earlier by Hallenbeck[3] via subordination methods. Subordination methods and extreme point methods complement each other in considering extremal problems. For example subordination can be used to show that for all $f \in R_\gamma$,

$$\int_0^{2\pi} |f'(re^{i\theta})|^p\, d\theta \leq \int_0^{2\pi} |k'_\alpha(re^{i\theta})|^p\, d\theta\,, \quad 0 < p < \infty\,,$$

while extreme point methods can be used to obtain

$$\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \leq \int_0^{2\pi} |k_\alpha(re^{i\theta})|^p\, d\theta\,, \quad 1 \leq p < \infty\,.$$

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