SOLUTIONS

1. If $y = x^2(4 + e^{-2x})$, find the differential of y when $x = 1$ and $dx = \frac{1}{2}$.

A. $dy = 4 + e^{-2}$

B. $dy = \frac{1}{2}(4+e^{-2})$

$\boxed{\textbf{C.}}$ $dy = 4$

D. $dy = \frac{1}{2}$

E. $dy = 4 + 2e^{-2}$

$$\frac{dy}{dx} = x^2(-2e^{-2x}) + 2x(4 + e^{-2x})$$

$$= -2x^2e^{-2x} + 8x + 2xe^{-2x}$$

$$\therefore \quad dy = (-2x^2e^{-2x} + 8x + 2xe^{-2x})\,dx$$

@ $x = 1$, $dx = \frac{1}{2}$

$$\Rightarrow dy = (-2e^{-2} + 8 + 2e^{-2})\left(\frac{1}{2}\right) = 4$$

2. Using linear approximation, approximate the value of $\sqrt{24.2}$.

A. 4.91

$\boxed{\textbf{B.}}$ 4.92

C. 4.94

D. 4.96

E. 5.80

$f(x) = \sqrt{x}$, $\quad a = 25$

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = f(25) + f'(25)(x-25)$$

$$L(x) = 5 + \frac{1}{2(5)}(x-25) = 5 + \frac{1}{10}(x-25)$$

$$f(x) \approx L(x)$$

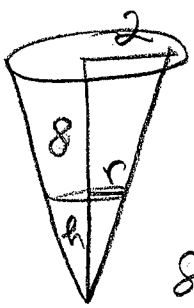
$$\therefore \quad f(24.2) \approx L(24.2) = 5 + \frac{1}{10}(24.2-25)$$

$$= 5 + \frac{1}{10}(-0.8)$$

$$= 5 + (-0.08)$$

$$= 4.92$$

3. A tank has the shape of an inverted circular cone with radius 2 ft and height 8 ft. If water is poured into the tank at a rate of 5 ft/min, find the rate at which the volume of water is increasing when the water is 4 ft deep. (Recall, $V = \frac{1}{3}\pi r^2 h$)



Given rate: $\frac{dh}{dt} = 5$ ft/min

Desired rate: $\frac{dV}{dt}$ when $h = 4$

A. 4π ft^3/min

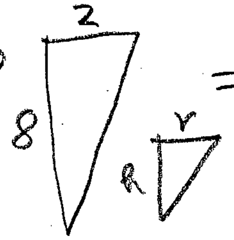
$\boxed{\text{B.}}$ 5π ft^3/min

C. 6π ft^3/min

D. $\frac{32\pi}{3}$ ft^3/min

E. 20π ft^3/min

Similar triangles $\Rightarrow$



$\Rightarrow \frac{2}{8} = \frac{r}{h}$

$\frac{h}{4} = r$

Relation: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{4}\right)^2 h \quad \therefore \quad V = \frac{\pi}{48} h^3$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{h=4} = \left.\frac{\pi}{48}(3h^2)\frac{dh}{dt}\right|_{h=4} = \left(\frac{\pi}{48}\right)\left(3(4)^2\right)(5) = \underline{\underline{5\pi}}$$

4. Let $f(x) = x + \dfrac{4}{x}$. If M = absolute maximum value of f and m = absolute mininum value of f over the closed interval $[1, 4]$, then the product $Mm = ?$

A. 0

B. 10

$\boxed{\text{C.}}$ 20

D. 25

E. 100

$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = 0$ when $x = \pm 2$

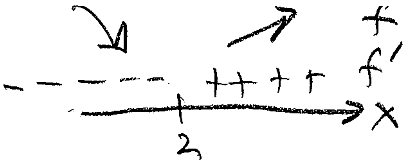
$x = 2$ only admissible critical number

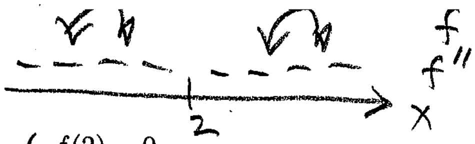
Table:

	x	$f(x) = x + \frac{4}{x}$	
Critical # $\rightarrow$	2	4	$\leftarrow m$
End points	1	5	$\leftarrow M$
	4	5	

$\therefore Mm = (5)(4)$

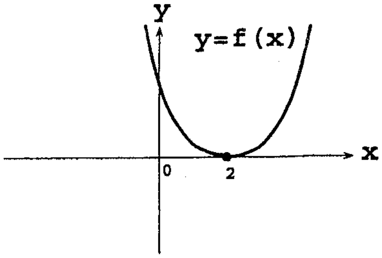
$= 20$



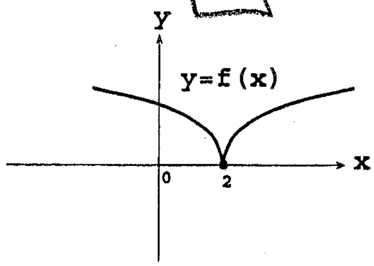


5. Which graph below satisfies these conditions: $\begin{cases} f(2) = 0 \\ f'(x) < 0, & \text{when } x < 2 \\ f'(x) > 0, & \text{when } x > 2 \\ f''(x) < 0, & \text{when } x < 2 \text{ and } x > 2 \end{cases}$

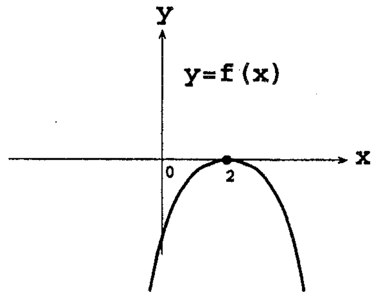
A.



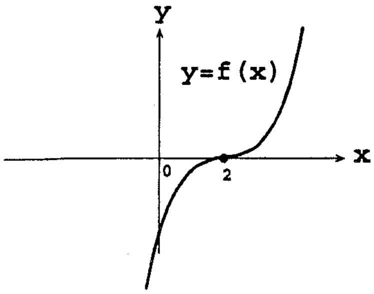
$\boxed{\textbf{B.}}$



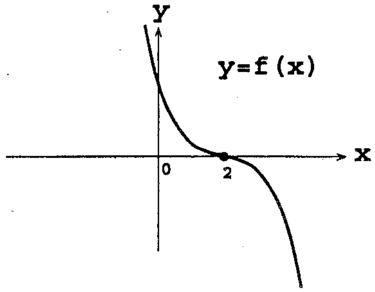
C.



D.

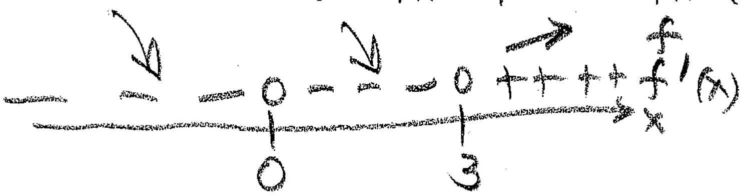


E.



6. The function $f(x) = x^4 - 4x^3 + 5$ is both *decreasing* and also *concave down* on which open interval below?

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ ✓



A. $(0, 3)$

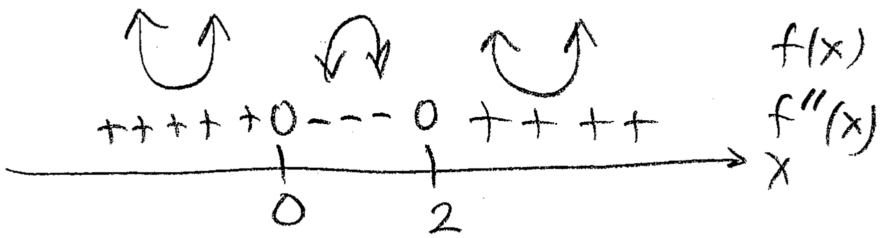
B. $(-\infty, 3)$

C. $(-\infty, 2)$

$\boxed{\textbf{D.}}$ $(0, 2)$

E. $(-\infty, 0)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$ ✓



7. If f is continuous on $[1, 6]$ and differentiable on $(1, 6)$ with $f(6) = 13$ and $f'(x) \geq 2$ for $1 < x < 6$, what is the largest possible value for $f(1)$?

MVT $\Rightarrow$ $f(b) - f(a) = f'(c)(b-a)$, $c \in (a, b)$

$\therefore$ $f(6) - f(1) = f'(c)(6-1)$

$13 - f(1) = 5f'(c) \geq 5(2) = 10$

$13 - f(1) \geq 10$

$13 - 10 \geq f(1)$

$3 \geq f(1)$

A. 2

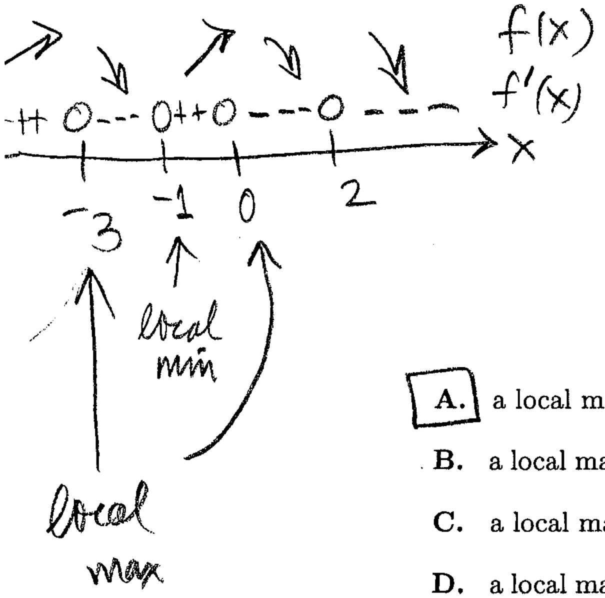
$\boxed{\text{B.}}$ 3

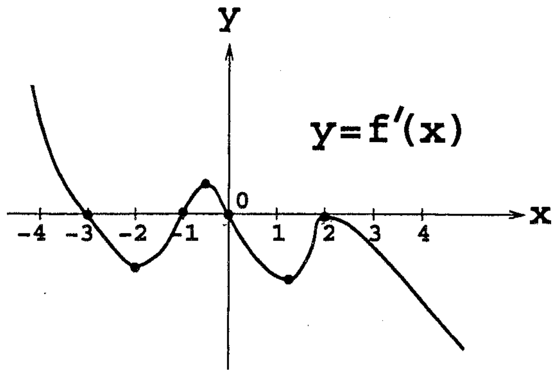
C. 8

D. 11

E. 13

8. Given the graph of $f'(x)$, the derivative of $f(x)$, shown below, then the function $f(x)$ has





$\boxed{\text{A.}}$ a local max at $x = -3$ and $x = 0$; a local min at $x = -1$

B. a local max at $x = -3$ and $x = 2$; a local min at $x = 1$ and $x = -2$

C. a local max at $x = \frac{1}{2}$; a local min at $x = -2$ and $x = 1$

D. a local max at $x = -3$ and $x = -1$; a local min at $x = 1$ and $x = 2$

E. a local max at $x = -1$ and $x = 0$; a local min at $x = -3$

9. Given that f is a differentiable function and the table below, which statement(s) is/are **TRUE**?

T **(I)** f has a local maximum value at $x = 0$

F **(II)** f has a local maximum value at $x = 2$

T **(III)** f is decreasing at $x = 1$

x	$f(x)$	$f'(x)$	$f''(x)$
0	-2	0	-6
1	3	-2	4
2	5	0	12

$f'(0) = 0$ and $f''(0) = -6 \Rightarrow f(0)$ local max value

$f'(2) = 0$ and $f''(2) = 12 \Rightarrow f(2)$ local min value

$f'(1) = -2 < 0 \Rightarrow f \searrow$

A. Only (I)

B. Only (III)

C. Only (II) and (III)

D. Only (I) and (II)

$\boxed{\textbf{E.}}$ Only (I) and (III)

10. Evaluate this limit: $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{3 - 3\sin x}{(x - \frac{\pi}{2})^2}$. $\overset{LR}{=} \displaystyle\lim_{x\to\frac{\pi}{2}} \frac{-3\cos x}{2(x - \frac{\pi}{2})}$

also Type $\frac{0}{0}$

Type $\frac{0}{0}$

$\overset{LR}{=} \displaystyle\lim_{x\to\frac{\pi}{2}} \frac{3\sin x}{2} = \frac{3}{2}$

A. 0

B. $-\frac{3}{4}$

C. 3

$\boxed{\textbf{D.}}$ $\frac{3}{2}$

E. $\frac{3}{4}$

$f'(x) = 2x+3$

11. If $f(x) = x^2 + 3x$ and $[a, b] = [-1, 2]$, find a number c in the interval $(-1, 2)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)\,.$$

$$\frac{f(2) - f(-1)}{2-(-1)} = 2c+3$$

$$\frac{10-(-2)}{3} = 2c+3$$

$$\frac{12}{3} = 2c+3$$

$$4 = 2c+3$$

$$\frac{1}{2} = c$$

A. $c = 0$

$\boxed{\text{B.}}$ $c = \frac{1}{2}$

C. $c = \frac{3}{2}$

D. $c = -\frac{1}{2}$

E. $c = 1$

12. If $\alpha = \lim_{x \to 0^+} (1 + 2x)^{\frac{3}{x}}$ and $\beta = \lim_{x \to \infty} \frac{2x + 3}{x^2 + 3x}$, then

Type 1^∞ Type $\frac{\infty}{\infty}$

A. $\alpha = 1, \quad \beta = 0$

B. $\alpha = e^{-3}, \quad \beta = 0$

C. $\alpha = e^6, \quad \beta = \infty$

D. $\alpha = e^3, \quad \beta = 1$

$\boxed{\text{E.}}$ $\alpha = e^6, \quad \beta = 0$

$$\alpha = \lim_{x \to 0^+} (1+2x)^{\frac{3}{x}} = \lim_{x \to 0^+} e^{\ln\left[(1+2x)^{\frac{3}{x}}\right]}$$

$$= \lim_{x \to 0^+} e^{\frac{3\ln(1+2x)}{x}} = e^{\lim_{x \to 0^+} \frac{3\ln(1+2x)}{x}}$$ Type $\frac{0}{0}$

$$\overset{LR}{=} e^{\lim_{x \to 0^+} \frac{3\left(\frac{2}{1+2x}\right)}{1}} = e^6$$ ✓

$$\beta = \lim_{x \to \infty} \frac{2x+3}{x^2+3x} \overset{LR}{=} \lim_{x \to \infty} \frac{2}{2x} = 0$$