Review - Quiz # 3

(1) Spring-Mass Systems $\begin{cases} m\,x'' + c\,x' + k\,x = F(t) \\ x(0) = x_0\,,\ x'(0) = x_1 \end{cases}$

$m =$ mass of object , $c =$ damping constant , $k =$ spring constant
Weight $w = m\,g$, Hooke's Law : $F_s = k\,d$, $F(t) =$ external force

A. Undamped Free Vibrations : $m\,x'' + k\,x = 0$ (Simple Harmonic Motion)
Note that $a\cos\omega t + b\sin\omega t = A\cos(\omega t - \phi)$, where $A = \sqrt{a^2+b^2} =$ *amplitude*, $\omega =$ *frequency*, $\dfrac{2\pi}{\omega} =$ *period* and $\phi =$ *phase shift* (Note: $\phi = \cos^{-1}\left\{\dfrac{a}{A}\right\}$ or $\phi = -\cos^{-1}\left\{\dfrac{a}{A}\right\}$, it depends on which quadrant the angle ϕ lies in).

B. Damped Free Vibrations : $m\,x'' + c\,x' + k\,x = 0$

(i) $c^2 - 4km > 0$ (*overdamped*) $\Rightarrow$ distinct real roots to CE

(ii) $c^2 - 4km = 0$ (*critically damped*) $\Rightarrow$ repeated roots to CE

(iii) $c^2 - 4km < 0$ (*underdamped*) $\Rightarrow$ complex roots to CE (motion is oscillatory)

C. Forced Vibrations : ($F(t) = F_0\cos\omega t$ or $F(t) = F_0\sin\omega t$, for example)

(i) $m\,x'' + k\,x = F_0\cos\omega t$ (no damping) If $\omega = \omega_0 = \sqrt{\dfrac{k}{m}}$ $\Rightarrow$ Resonance occurs and the solution is unbounded; while if $\omega \neq \omega_0$ then motion is a series of *beats* (solution is bounded)

(ii) $m\,x'' + c\,x' + k\,x = F(t)$ (damping) If you write the general solution as $x(t) = x_h(t) + x_p(t)$, then $x_h(t) =$ *Transient Solution* ; $x_p(t) =$ *Steady-State Solution.* (Note: transient solution satisfies $x_h(t) \longrightarrow 0$ as $t \longrightarrow \infty$)

(2) Systems of Linear Differential Equations $\vec{\mathbf{X}}\,'(t) = A\,\vec{\mathbf{X}}(t)$

A. Should be able to rewrite a single n^{th} order equation as a system of 1^{st} order equations.

B. Should be able to solve 2×2 systems of 1^{st} order equations $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

using the **Elimination Method** and using *Eigenvalues* and *Eigenvectors* :

Eigenvalues : roots to $|A - \lambda\,I| = \begin{vmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{vmatrix} = 0$

Eigenvector : $\vec{\mathbf{v}}$ is a nonzero solution to $(A - \lambda\,I)\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

C. If $\vec{\mathbf{x}}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}$ and if $\vec{\mathbf{x}}^{(2)}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$, the *Wronskian* is $W = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$.
If $\vec{\mathbf{x}}^{(1)}(t)$ and $\vec{\mathbf{x}}^{(2)}(t)$ are linearly independent solutions of $\vec{\mathbf{X}}\,'(t) = A\,\vec{\mathbf{X}}(t)$, then a *Fundamental matrix* of the system is $\Phi(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}$.

(3) **Undetermined Coefficients for Systems** : Want to find a particular solution, $\vec{\mathbf{x}}_p(t)$, of

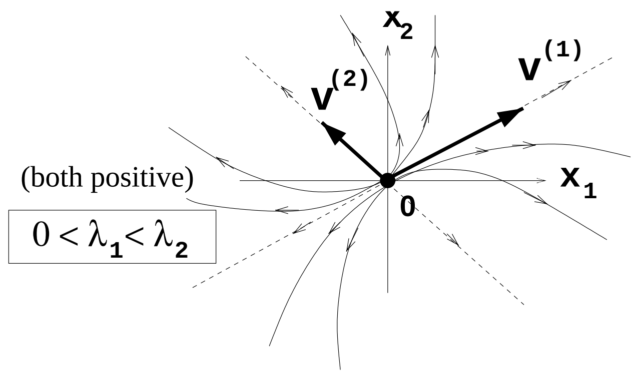
$$\vec{\mathbf{X}}\,'(t) = \mathbf{A}\,\vec{\mathbf{X}}(t) + \vec{\mathbf{F}}(t)$$

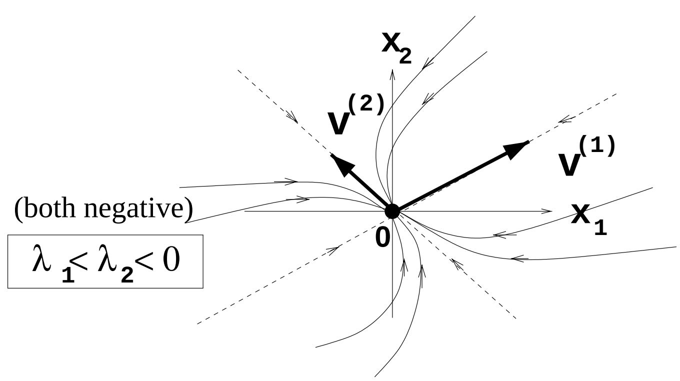
The column vector $\vec{\mathbf{F}}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ must have component functions $f_1(t)$ and $f_2(t)$ as one of the three special forms like those for Undetermined Coefficients for regular 2^{nd} order equations (see Review Sheet # 2). The main difference is if $\vec{\mathbf{F}}(t) = \vec{\mathbf{u}}\,e^{\lambda t}$ and λ is also an eigenvalue of A, then try a particular solution of the form $\vec{\mathbf{x}}_p = \vec{\mathbf{a}}\,te^{\lambda t} + \vec{\mathbf{b}}\,e^{\lambda t}$.

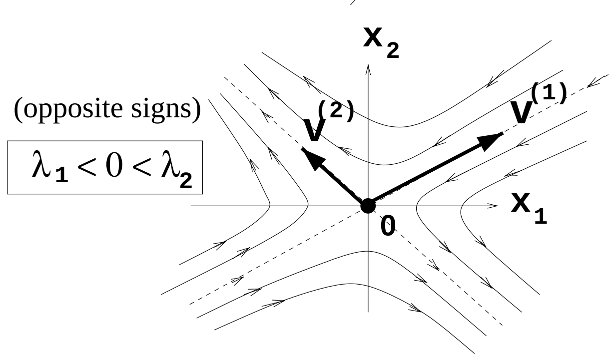
(4) **Mixing Problems (several tanks)** : Flow through several connected tanks leads to a system of linear 1^{st} order equations. You should be able to set up and solve such problems. (Draw a picture to help with the analysis.)

(5) **Phase Portraits** : A plot of the trajectories (solutions) of a given system $\vec{\mathbf{X}}\,'(t) = A\,\vec{\mathbf{X}}(t)$ is called a **phase portrait**. To sketch the phase portrait, we need to consider 3 cases :

(i) $\lambda_1 < \lambda_2$, real and distinct If $\vec{\mathbf{v}}^{(1)}$, $\vec{\mathbf{v}}^{(2)}$ are e-vectors corresponding to λ_1 and λ_2, respectively $\Rightarrow$ $\vec{\mathbf{x}}^{(1)}(t) = e^{\lambda_1 t}\,\vec{\mathbf{v}}^{(1)}$ and $\vec{\mathbf{x}}^{(2)}(t) = e^{\lambda_2 t}\,\vec{\mathbf{v}}^{(2)}$ are solutions and hence general solution is $\vec{\mathbf{x}}(t) = C_1\,\vec{\mathbf{x}}^{(1)}(t) + C_2\,\vec{\mathbf{x}}^{(2)}(t)$.

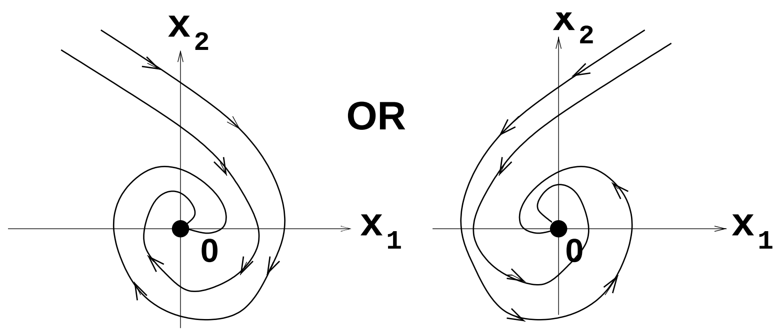






(ii) $\underline{\lambda_1 = \alpha + i\beta}$ If $\vec{\mathbf{w}} = \vec{\mathbf{a}} + i\,\vec{\mathbf{b}}$ is a complex e-vector corresponding to λ_1 then
$\vec{\mathbf{x}}^{(1)}(t) = \Re e\left\{e^{\lambda_1 t}\vec{\mathbf{w}}\right\} = e^{\alpha t}\,(\vec{\mathbf{a}}\cos\beta t - \vec{\mathbf{b}}\sin\beta t)$ and
$\vec{\mathbf{x}}^{(2)}(t) = \Im m\left\{e^{\lambda_1 t}\vec{\mathbf{w}}\right\} = e^{\alpha t}\,(\vec{\mathbf{a}}\sin\beta t + \vec{\mathbf{b}}\cos\beta t)$ are real-valued solutions and hence general solution is $\vec{\mathbf{x}}(t) = C_1\,\vec{\mathbf{x}}^{(1)}(t) + C_2\,\vec{\mathbf{x}}^{(2)}(t)$.

If $\alpha < 0$:



(Test a point to decide which)

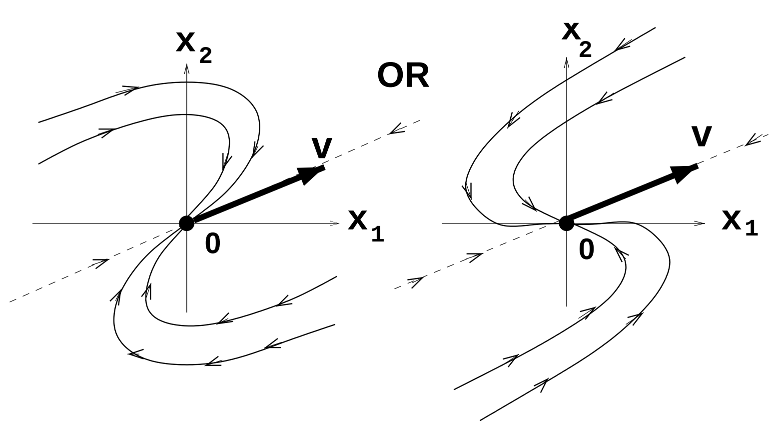
(iii) $\underline{\lambda_1 = \lambda_2}$ $\Rightarrow$ Solutions are $\vec{\mathbf{x}}^{(1)}(t) = e^{\lambda_1 t}\,\vec{\mathbf{v}}$ and $\vec{\mathbf{x}}^{(2)}(t) = t\,e^{\lambda_1 t}\,\vec{\mathbf{v}} + e^{\lambda_1 t}\,\vec{\mathbf{w}}$, where

$$(A - \lambda_1 I)\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$$
$$(A - \lambda_1 I)\,\vec{\mathbf{w}} = \vec{\mathbf{v}}$$

($\vec{\mathbf{v}}$ is an eigenvector while $\vec{\mathbf{w}}$ is a "generalized eigenvector")

The general solution of system is $\vec{\mathbf{x}}(t) = C_1\,\vec{\mathbf{x}}^{(1)}(t) + C_2\,\vec{\mathbf{x}}^{(2)}(t)$.

If $\lambda_1 < 0$:



(Test a point to decide which)

Practice Problems

1. For what nonnegative values of γ will the the solution of the initial value problem
$u'' + \gamma u' + 4u = 0,\ u(0) = 4,\ u'(0) = 0$ *oscillate* ?

2. (a) For what positive values of k does the solution of the initial value problem
$2u'' + ku = 3\cos(2t),\ u(0) = 0,\ u'(0) = 0$, become *unbounded* (Resonance) ?
(b) For what positive values of k does the solution of the initial value problem
$2u'' + u' + ku = 3\cos(2t),\ u(0) = 0,\ u'(0) = 0$, become *unbounded* (Resonance) ?

3. Find the steady–state solution of the IVP $y'' + 4y' + 4y = \sin t,\ y(0) = 0,\ y'(0) = 0$.

4. A 4-kg mass stretches a spring 0.392 m. If the mass is released from 1 m below the equilibrium position with a downward velocity of 10 m/sec, what is the maximum displacement ?

5. Use the Elimination Method to solve the system $\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{cases}$

6. Rewrite the 2^{nd} order differential equation $y'' + 2y' + 3ty = \cos t$ with $y(0) = 1, y'(0) = 4$ as a system of 1^{st} order differential equations.

7. Find eigenvalues and corresponding eigenvectors of (a) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$

8. Find the solution of the IVP $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
Find a fundamental matrix $\Phi(t)$.

9. Solve $\begin{pmatrix} x_1{}' \\ x_2{}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{\mathbf{x}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

10. Find the general solution of the system $\vec{\mathbf{x}}\,'(t) = A\,\vec{\mathbf{x}}(t)$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

11. Tank # 1 initially holds 50 gals of brine with a concentration of 1 lb/gal, while Tank # 2 initially holds 25 gals of brine with a concentration of 3 lb/gal. Pure H_2O flows into Tank # 1 at 5 gal/min. The well-stirred solution from Tank # 1 then flows into Tank # 2 at 5 gal/min . The solution in Tank # 2 flows out at 5 gal/min. Set up and solve an IVP that gives $x_1(t)$ and $x_2(t)$, the amount of salt in Tanks # 1 and # 2, respectively, at time t.

12. Tank # 1 initially holds 50 gals of brine with concentration of 1 lb/gal and Tank # 2 initially holds 25 gals of brine with concentration 3 lb/gal. The solution in Tank # 1 flows at 5 gal/min into Tank # 2, while the solution in Tank # 2 flows back into Tank # 1 at 5 gal/min. Set up an IVP that gives $x_1(t)$ and $x_2(t)$, the amount of salt in Tanks # 1 and # 2, respectively, at time t.

13. Find the general solution of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t$.

14. Find a particular solution of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

15. Find the general solution of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 6e^{-t} \\ 1 \end{pmatrix}$.

16. Match the phase portraits shown below that best corresponds to each of the given systems of differential equations:

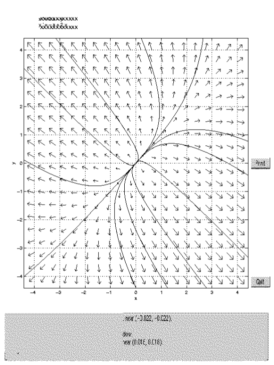
(i) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

(ii) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$

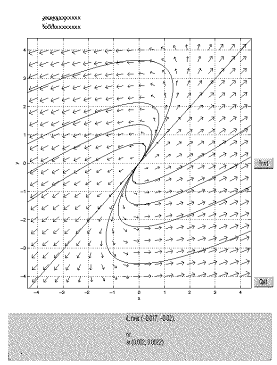
(iii) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{t} + C_2 e^{t} \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

(iv) $\vec{\mathbf{x}}\,' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}$

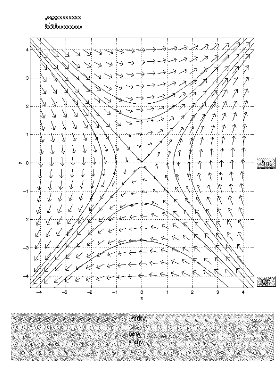
A.



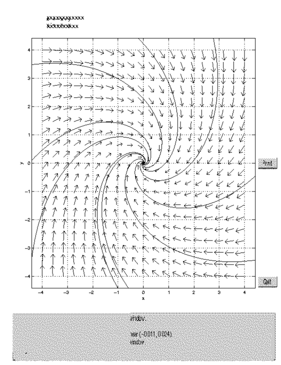
B.



C.



D.



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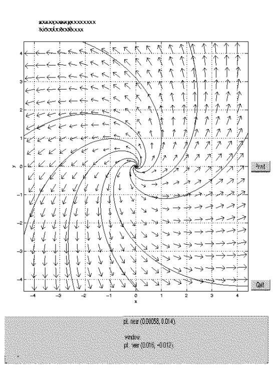
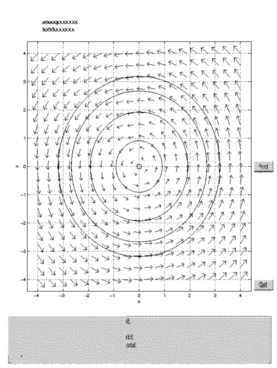
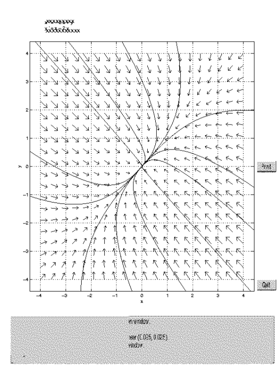


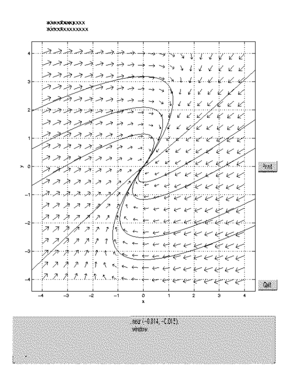
F.



G.



H.



ANSWERS

1. $0 \le \gamma < 4$ **2.** (a) $k = 8$ (resonance) (b) NO value of k, all solutions are bounded.

3. $y = \frac{1}{25}(3\sin t - 4\cos t)$ **4.** $x(t) = \cos 5t + 2\sin 5t = \sqrt{5}\cos(5t - \phi),\ \phi = \cos^{-1}\frac{1}{\sqrt{5}} \approx 1.1$ Thus amplitude $= \sqrt{5}$. **5.** $x_1(t) = C_1 e^{3t} + C_2 e^{-t},\ x_2(t) = 2C_1 e^{3t} - 2C_2 e^{-t}$

6. Let $x_1 = y,\ x_2 = y'$, then $\begin{cases} x_1' = x_2 \\ x_2' = -3tx_1 - 2x_2 + \cos t \end{cases}$, where $x_1(0) = 1,\ x_2(0) = 4$

7. (a) $\lambda_1 = 3,\ \vec{\mathbf{v}}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = -1,\ \vec{\mathbf{v}}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

7. (b) $\lambda_1 = -1,\ \vec{\mathbf{v}}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\lambda_2 = -2,\ \vec{\mathbf{v}}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

8. $\vec{\mathbf{x}}(t) = 2\,e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix},\quad \Phi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$

9. $\vec{\mathbf{x}}(t) = 2\,e^{t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} - e^{t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ **10.** $\vec{\mathbf{x}}(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2\left\{e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + te^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

11. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -\frac{1}{10} & 0 \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 75 \end{pmatrix}$

Solution : $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 50e^{-\frac{t}{10}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 25e^{-\frac{t}{5}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

12. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -\frac{1}{10} & \frac{1}{5} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 75 \end{pmatrix}$

Solution : $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{125}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{100}{3}\,e^{-\frac{3t}{10}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

13. $\vec{\mathbf{x}}(t) = C_1 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **14.** $\vec{\mathbf{x}}_p(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

15. $\vec{\mathbf{x}}(t) = C_1 e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **16.** (i) **C** (ii) **A** (iii) **B** (iv) **D**

Remark: There is an analogy between spring-mass systems and RLC circuits given by :

Spring-Mass System	RLC Circuit
$mx'' + cx' + kx = F(t)$	$LQ'' + RQ' + \frac{1}{C}Q = E(t)$
$x =$ Displacement	$Q =$ Charge
$x' =$ Velocity	$Q' = I =$ Current
$m =$ Mass	$L =$ Inductance
$c =$ Damping constant	$R =$ Resistance
$k =$ Spring constant	$\frac{1}{C} =$ (Capacitance)$^{-1}$
$F(t) =$ External force	$E(t) =$ Voltage