*********** Disclaimer ***********

This represents a very brief outline of most of the topics covered in this course. To be fully prepared you must read your class notes, the book and correctly work as many problems as possible.

CHAPTER 11

1. Vector arithmetic; directed vector $\overline{P_0P_1}$ from P_0 to P_1; dot product of vectors $(a_1\vec{\mathbf{i}}+a_2\vec{\mathbf{j}}+a_3\vec{\mathbf{k}})\cdot(b_1\vec{\mathbf{i}}+b_2\vec{\mathbf{j}}+b_3\vec{\mathbf{k}}) = a_1b_1+a_2b_2+a_3b_3$; angle between two vectors, $\cos\theta = \dfrac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{||\vec{\mathbf{a}}||\,||\vec{\mathbf{b}}||}$; cross product $\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ and their properties: $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, $\frac{1}{2}||\vec{\mathbf{a}}\times\vec{\mathbf{b}}|| =$ area of triangle spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$; projections $\boldsymbol{pr}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \left(\dfrac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{||\vec{\mathbf{a}}||^2}\right)\vec{\mathbf{a}}$; $\vec{\mathbf{v}} = ||\vec{\mathbf{v}}||(\cos\theta\,\vec{\mathbf{i}}+\sin\theta\,\vec{\mathbf{j}})$.

2. Equation of line containing (x_0, y_0, z_0), direction vector $\vec{\mathbf{L}} = a\vec{\mathbf{i}}+b\vec{\mathbf{j}}+c\vec{\mathbf{k}}$:

   (a) Vector Form: $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\,\vec{\mathbf{L}}$, where $\vec{\mathbf{r}}_0 = x_0\vec{\mathbf{i}}+y_0\vec{\mathbf{j}}+z_0\vec{\mathbf{k}}$

   (b) Parametric Form: $\begin{cases} x = x_0 + a\,t \\ y = y_0 + b\,t \\ z = z_0 + c\,t \end{cases}$

   (c) Symmetric Form: $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$
   (if say $b = 0$, then $\frac{x-x_0}{a} = \frac{z-z_0}{c}$; $y = y_0$)

3. Equation of plane containing (x_0, y_0, z_0), normal vector $\vec{\mathbf{N}} = a\vec{\mathbf{i}}+b\vec{\mathbf{j}}+c\vec{\mathbf{k}}$:

$$\vec{\mathbf{N}}\cdot\overline{\mathbf{P_0P}} = 0 \quad \text{or} \quad a(x-x_0)+b(y-y_0)+c(z-z_0) = 0\,.$$

4. Sketching planes (look at intercepts : $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$).

CHAPTER 12

1. Differentiating and integrating vector-valued functions and sketching the corresponding curves.

2. Parameterizing curves of the form say $y = f(x),\ a \le x \le b$ $(C:\ \vec{\mathbf{r}}(t) = t\,\vec{\mathbf{i}} + f(t)\,\vec{\mathbf{j}},\ a \le t \le b)$.

3. Unit tangent vector $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}'(t)}{||\vec{\mathbf{r}}'(t)||}$; length of a curve $\displaystyle\int_a^b ||\vec{\mathbf{r}}'(t)||\,dt$.

CHAPTER 13

1. Domains of functions of several variables; level curves $f(x, y) = C$, level surfaces $f(x, y, z) = C$; sketching surfaces using level curves.

2. Quadric surfaces.

3. Computing limits, determining when limits exist.

4. Partial derivatives; CHAIN RULE (consider tree diagrams).

5. Implicit Differentiation, for example :

   (a) If $y = y(x)$ is defined implicitly by $F(x, y) = 0$, then $\dfrac{dy}{dx} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

   (b) If $z = z(x, y)$ is defined implicitly by $F(x, y, z) = 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\,.$$

6. Gradients: $\nabla f(x, y, z) = f_x\,\vec{\mathbf{i}} + f_y\,\vec{\mathbf{j}} + f_z\,\vec{\mathbf{k}}$; the gradient $\nabla f(x, y)$ is perpendicular to level curve $f(x, y) = C$ and $\nabla f(x, y, z)$ is perpendicular to level surface $f(x, y, z) = C$.

7. Directional derivative : $D_{\vec{\mathbf{u}}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ is a UNIT vector; $-||\nabla f|| \le D_{\vec{\mathbf{u}}} f \le ||\nabla f||$;
   $f(x, y, z)$ increases fastest in the direction ∇f.

8. Normal vector $\vec{\mathbf{n}}$ to surfaces $\sum$:

   (a) $\sum$ is a level surface, $F(x, y, z) = C$, then a normal is $\vec{\mathbf{n}} = \nabla F(x, y, z)$.

   (b) $\sum$ is the graph of $z = f(x, y)$, then a normal is $\vec{\mathbf{n}} = -f_x\,\vec{\mathbf{i}} - f_y\,\vec{\mathbf{j}} + \vec{\mathbf{k}}$

9. Tangent planes to surfaces; Tangent Plane Approximation Formula:

$$f(x + h, y + k) \approx f(x, y) + f_x(x, y)\,h + f_y(x, y)\,k\,.$$

10. Critical points of $f(x, y, z)$: points where $\nabla f(x, y, z) = \vec{\mathbf{0}}$ or $\nabla f(x, y, z)$ does not exist.

11. Finding relative extrema of $f(x,y)$ at those particular critical points (x_0, y_0) where $\nabla f(x_0,y_0) = \vec{\mathbf{0}}$ using 2^{nd} Partials Test: let $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

    (a) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0 \Rightarrow f$ has rel minimum value at (x_0,y_0)

    (b) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0 \Rightarrow f$ has rel maximum value at (x_0,y_0)

    (c) If $D(x_0,y_0) < 0 \Rightarrow f$ has a saddle point at (x_0,y_0).

12. Finding absolute extrema over closed, bounded regions: find interior critical points, find points on the boundary where extrema may occur, make a table of values of f at all these points.

13. Constrained extremal problems: Maximize and/or minimize $f(x,y)$ subject to the condition $g(x,y) = C$; Lagrange Multipliers: $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = C \end{cases}$

Chapter 14 (DRAW PICTURES FOR THIS CHAPTER)

1. Double integrals; vertically and horizontally simple regions, iterated integrals; double integrals in polar coordinates ($dA = r\,dr\,d\theta$)

2. Applications of double integrals: areas between curves, volumes, surface area $S = \int\int_R \sqrt{f_x^2 + f_y^2 + 1}\; dA$.

3. Changing the order of integration in double integrals.

4. Triple integrals; iterated triple integrals; applications of triple integrals: volumes, mass $m = \int\int\int_D \delta(x,y,z)\,dV$.

5. Triple integrals in Rectangular, Cylindrical, and Spherical Coordinates:

    (a) Rectangular Coordinates: $dV = dz\,dy\,dx$ or $dV = dz\,dx\,dy$, etc

    (b) Cylindrical Coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$ $dV = r\,dz\,dr\,d\theta$

    (c) Spherical Coordinates: $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$ $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Chapter 15 (DRAW PICTURES FOR THIS CHAPTER)

1. Vector fields $\vec{\mathbf{F}} = M\,\vec{\mathbf{i}} + N\,\vec{\mathbf{j}} + P\,\vec{\mathbf{k}}$; divergence and curl of a vector field $\vec{\mathbf{F}}$:

$$\operatorname{div}\vec{\mathbf{F}} = \nabla\cdot\vec{\mathbf{F}} = M_x + N_y + P_z$$

$$\operatorname{curl}\vec{\mathbf{F}} = \nabla\times\vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix};$$

   Laplacian of $f = \operatorname{div}\nabla f = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$.

2. Conservative vector fields $\vec{\mathbf{F}} = \nabla f$; how to determine if $\vec{\mathbf{F}}$ is conservative : check that $\operatorname{curl}\vec{\mathbf{F}} = \vec{\mathbf{0}}$ (if region has no "holes"); given that $\vec{\mathbf{F}} = \nabla f$, know how to determine the potential function $f(x,y,z)$.

3. Line integrals of functions $\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,||\vec{\mathbf{r}}\,'(t)||\,dt$; line integrals of vector fields $\vec{\mathbf{F}} = M\,\vec{\mathbf{i}} + N\,\vec{\mathbf{j}} + P\,\vec{\mathbf{k}}$:

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}\,'(t)\,dt$$

   or equivalently $\int_C M\,dx + N\,dy + P\,dz = \int_a^b Mx'dt + Ny'dt + Pz'dt$, where $C:\ \vec{\mathbf{r}}(t) = x(t)\,\vec{\mathbf{i}} + y(t)\,\vec{\mathbf{j}} + z(t)\,\vec{\mathbf{k}},\ \ a \le t \le b$.

4. Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot\vec{\mathbf{dr}} = f(P_1) - f(P_0)$; independence of path (check if $\vec{\mathbf{F}} = \nabla f$ or $\operatorname{curl}\vec{\mathbf{F}} = \vec{\mathbf{0}}$) ; applications to work $W = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}$.

5. <u>Green's Theorem</u> : If C is a closed curve traversed counterclockwise, then

$$\int_C M(x,y)\,dx + N(x,y)\,dy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\,.$$

6. Surface integrals : if $\sum$ is the graph of $z = f(x,y)$ with $(x,y)\in R$, then $\int\int_{\sum} g(x,y,z)\,dS = \int\int_R g(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\ dA\,.$

7. Flux integral of $\vec{\mathbf{F}} = M\,\vec{\mathbf{i}} + N\,\vec{\mathbf{j}} + P\,\vec{\mathbf{k}}$ over the surface $\sum$, the graph of $z = f(x, y)$ with $(x, y) \in R$, and $\vec{\mathbf{n}} =$ upper unit normal vector to $\sum$:

$$\int\!\!\int_{\sum} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\; dS = \int\!\!\int_{R} \{-M\, f_x - N\, f_y + P\}\; dA\,.$$

8. <u>DIVERGENCE THEOREM (GAUSS' THEOREM)</u> : If D is a solid region and $\sum$ is its closed boundary surface, $\vec{\mathbf{n}} =$ outer unit normal to $\sum$, then

$$\int\!\!\int_{\sum} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\; dS = \int\!\!\int\!\!\int_{D} \operatorname{div} \vec{\mathbf{F}}\; dV\,.$$