Solutions

1. \[ \int_{\gamma_R} \frac{e^{iz}}{z^2 + \alpha^2} \, dz = \int_{\gamma_R} \frac{\frac{e^{iz}}{z + \alpha i}}{z - \alpha i} \, dz \]

\[ = 2\pi i \left\{ \frac{e^{iz}}{z + \alpha i} \right\}_{z = \alpha i} = 2\pi i \frac{e^{-\alpha}}{2\alpha i} = \frac{\pi}{\alpha e^{\alpha}} \]

#2

Page 161 #10:
\[ \int_{\gamma} \frac{f'(z)}{z - z_0} \, dz = \int_{\gamma} \frac{f(z)}{(z - z_0)^2} \, dz \] (x)

Case 1: \( z_0 \) outside \( \gamma \). Then LHS of (x) is zero
and RHS of (x) is zero (both by Cauchy's Thm)

Case 2: \( z_0 \) inside \( \gamma \). Then since \( f' \) is also analytic inside and on \( \gamma \), by the C1F, the LHS of (x) = \( 2\pi i f'(z_0) \)

By the C1FFD, the RHS of (x) = \( \frac{2\pi i}{1!} f'(z_0) \)

Hence (x) holds for all \( z_0 \) not on \( \gamma \)
Consider the circle $C_r: |z - z_0| = r$.

By the Cauchy Integral Formula (CIF),

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z - z_0} \, dz.$$

Now parameterize $C_r: z(\theta) = z_0 + re^{i\theta}, \, 0 \leq \theta \leq 2\pi$.

Then

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \{i re^{i\theta}\} \, d\theta$$

i.e.,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta.$$
If \( f(z) = 10 \) on \( |z| = 1 \) and \( f \) is analytic inside and on \( |z| = 1 \), show that \( f(z) = 10 \) for all \( |z| < 1 \).

**Proof:** Let \( z_0 \) be an arbitrary point inside \( |z| = 1 \):

By the CIF,

\[
\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_{|z| = 1} \frac{f(z)}{z-z_0} \, dz = \frac{1}{2\pi i} \int_{|z| = 1} \frac{10}{z-z_0} \, dz = 10.
\]
#5. If $f$ is entire and $|f(z)| \leq 100$ for $|z| \geq 3$, show that $f(z) = C$.

**Proof #1:** We know $|f(z)| \leq 100$ for $|z| \geq 3$.

In particular, $|f(z)| \leq 100$ for $|z| = 3$.

By the Maximum-Modulus Principle, we must have $|f(z)| \leq 100$ for $|z| \leq 3$ (unless it is a constant function, in which case we're done).

Thus $|f(z)| \leq 100$ for all $z$.

Hence $f$ is entire and bounded, so by Liouville's Theorem, it must be a constant $f(z) = C$.

Here is another proof, just using the [C]FF [D]D:

(Cont'd)
Proof 2: It suffices to show $f'(z) = 0$ for every $z \in \mathbb{C}$.

Let $z_0$ be an arbitrary point in $\mathbb{C}$:

Let $C_R$ be the circle $|z| = R$ with $R > 3$.

Then $|f(z)| \leq 100$ for all $z$ on $C_R$.

Thus by Cauchy's Integral Formula (CIFD),

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^2} \, dz$$

Now for any $z \in C_R$, we have

$$\left| \frac{f(z)}{(z-z_0)^2} \right| \leq \frac{100}{(12-|z_0|)^2}$$

So

$$\left| \frac{f(z)}{(z-z_0)^2} \right| \leq \frac{100}{(R-12)^2}$$

By ML inequality, $|f'(z_0)| \leq \frac{100}{(R-12)^2} (2\pi R) \to 0$ as $R \to \infty$

$\therefore f'(z_0) = 0$. 