PROOF OF THE SENDOV CONJECTURE FOR POLYNOMIALS OF DEGREE AT MOST EIGHT

by

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Abstract. The well-known Sendov Conjecture asserts that if all the zeros of a polynomial p lie in the closed unit disk then there must be a critical point of p within unit distance of each zero. A method is presented which proves this conjecture for polynomials of degree $n \leq 8$ or for arbitrary degree n if there are at most eight distinct zeros.

1. Introduction

If p is a polynomial then the Gauss-Lucas Theorem states that all the critical points of p lie in the closed convex hull of its zeros. The Sendov Conjecture involves the location of critical points relative to each individual zero. More precisely:

Sendov Conjecture. *If $p(z) = \prod_{k=1}^{n} (z - z_k)$ is a polynomial with all its zeros inside the closed unit disk, then each of the disks $|z - z_k| \leq 1$, $k = 1, 2, \ldots, n$, must contain a zero of p'.*

The constant "1" is best possible upon considering $p(z) = z^n - 1$ (this and its rotations are suspected extremal polynomials). This conjecture (also known as Illief's Conjecture) has been open since appearing in Hayman's *Research Problems in Function Theory* [8, Problem 4.5] in 1967. It has been verified for $n = 3$ ([4]; 1968), $n = 4$ ([13]; 1968), $n = 5$ ([12]; 1969) and, after a quarter century, for $n = 6$ ([9], [2]; 1994) and $n = 7$ ([3] 1996 ;[7] 1997). It has also been verified for some special classes of polynomials (see Schmeisser [15]). The proofs for $n = 5, 6$, and 7 were obtained through slightly different estimates with some involved computations. We present here a unified method for investigating the Sendov Conjecture. As an application, we prove the conjecture for polynomials of degree $n \leq 8$ and identify all extremal polynomials:

Theorem 1.1. *If $p(z) = \prod_{k=1}^{n} (z - z_k)$, $|z_k| \leq 1$, $k = 1, 2, \ldots, n$ and $n = 2, 3, \ldots 8$, then each disk $|z - z_k| \leq 1$ $(k = 1, 2, \ldots, n)$ contains a zero of p'.*

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Corollary 1.1. *The only extremal polynomials for the Sendov Conjecture for* $n = 2, 3, \ldots, 8$ *have the form* $p(z) = z^n - e^{i\gamma}$, *where* $\gamma \in \mathbb{R}$.

The technique used here to prove these results is based on obtaining good upper and lower estimates on the product of the moduli of the critical points of p.

2. Known Results

Let $\mathcal{P}_n$ denote the set of all monic polynomials of degree n of the form

$$p(z) = \prod_{k=1}^{n} (z - z_k), \quad |z_k| \leq 1 \ (k = 1, 2, \ldots, n)$$

with

$$p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j).$$

If we define $I(z_k) = \min_{1 \leq j \leq n-1} |z_k - \zeta_j|, I(p) = \max_{1 \leq k \leq n} I(z_k)$, and $I(\mathcal{P}_n) = \sup_{p \in \mathcal{P}_n} I(p)$, then the Sendov Conjecture asserts that $I(\mathcal{P}_n) = 1$. (Since $z^n - 1 \in \mathcal{P}_n$, we know $I(\mathcal{P}_n) \geq 1$). The Gauss-Lucas Theorem gives $I(\mathcal{P}_n) \leq 2$. The best upper bound was given by Bojanov, Rahman and Szynal [1] who showed that $I(\mathcal{P}_n) \leq 1.0833\cdots$ and that $I(\mathcal{P}_n) \to 1$ as $n \to \infty$. It was proved in [13] that there exists an extremal polynomial p_n^* for each $n \geq 2$, i.e., $I(\mathcal{P}_n) = I(p_n^*) = I(z_{j_0})$ and that p_n^* has a zero on each closed subarc of $|z| = 1$ of length π. It will suffice to prove the Sendov Conjecture assuming p is an extremal polynomial. By a rotation, if necessary, we may thus suppose that $p \in \mathcal{P}_n$ and has the form

$$p(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k), \tag{2.1}$$

with $0 \leq a \leq 1$ and $I(\mathcal{P}_n) = I(p) = I(a)$. If $a = 0$ then $I(a) < 1$, hence p cannot be extremal. The case $a = 1$ is covered in the result of Rubinstein:

Lemma A [14]**.** *If* $p \in \mathcal{P}_n$ *and* $|z_{k_0}| = 1$, *then* $I(z_{k_0}) \leq 1$ *and equality occurs only for* $p(z) = z^n - e^{i\gamma}$, *where* $\gamma \in \mathbb{R}$.

Since $p'(a) = q(a)$ and $\dfrac{p''(a)}{p'(a)} = \dfrac{2q'(a)}{q(a)}$, where $q(z) = \dfrac{p(z)}{z-a}$, we have

$$n \prod_{j=1}^{n-1} (a - \zeta_j) = \prod_{k=1}^{n-1} (a - z_k) \tag{2.2}$$

and

$$\sum_{j=1}^{n-1} \frac{1}{a - \zeta_j} = \sum_{k=1}^{n-1} \frac{2}{a - z_k}. \tag{2.3}$$

Let $r_k = |a - z_k|$ and $\rho_j = |a - \zeta_j|$, for $j, k = 1, 2, \ldots n-1$. By relabeling we will suppose that

$$\rho_1 \leq \rho_j, \quad j = 1, 2, \ldots, n-1$$

It is known (see for example [11]) that

$$2\rho_1 \sin \frac{\pi}{n} \leq r_k \leq 1 + a, \quad k = 1, 2, \ldots, n-1. \tag{2.4}$$

If $a \neq 0$ is real and w a complex number with $w \neq a$, then a useful identity is

$$\operatorname{Re} \left\{ \frac{1}{a - w} \right\} = \frac{1}{2a} - \frac{|w|^2 - a^2}{2a|a - w|^2}. \tag{2.5}$$

In view of this identity and (2.3), we will need estimates on $\sum_{k=1}^{n-1} \frac{1}{r_k^2}$ which will be important later. To this end we will use:

Lemma B [12]. *If $r_1, r_2, \ldots, r_N, m, M$ and C are positive constants with $m \leq r_k \leq M$, $\prod_{k=1}^{N} r_k \geq C$ and $m^N \leq C \leq M^N$, then*

$$\sum_{k=1}^{N} \frac{1}{r_k^2} \leq \frac{N - \nu}{m^2} + \frac{\nu - 1}{M^2} + \left\{ \frac{m^{N-\nu} M^{\nu-1}}{C} \right\}^2$$

where $\nu = \min\{j \in \mathbb{Z} : M^j m^{N-j} \geq C\}$.

(*Note that* $\nu = \left[\left[\frac{\log\left(\frac{C}{m^N}\right)}{\log\left(\frac{M}{m}\right)} \right]\right]$, *where* $[[x]]=$ *smallest integer* $\geq x$.)

Define $a_n(\nu)$ and $S_n(a, \nu)$ for $\nu = 1, 2, \ldots, n-1$ as

$$a_n(\nu) \equiv \left[\frac{n}{(2 \sin \frac{\pi}{n})^{n-1-\nu}} \right]^{\frac{1}{\nu}} - 1 \tag{2.6}$$

and

$$S_n(a, \nu) \equiv \frac{(n-1-\nu)}{(2 \sin \frac{\pi}{n})^2} + \frac{(\nu - 1)}{(1+a)^2} + \left[\frac{(2 \sin \frac{\pi}{n})^{n-1-\nu} (1+a)^{\nu-1}}{n} \right]^2. \tag{2.7}$$

Note that for $n \geq 4$ and $\nu = 2, 3, \ldots, n-1$, we have $a_n(\nu) < a_n(\nu - 1)$. If $\rho_1 \geq 1$, then $2 \sin \frac{\pi}{n} \leq r_k \leq 1 + a$ and (2.2) implies that $\prod_{k=1}^{n-1} r_k \geq n$. Apply Lemma B to get the estimate

$$\mu_n(a) \equiv \sum_{k=1}^{n-1} \frac{1}{r_k^2} \leq S_n(a, \nu), \quad a_n(\nu) \leq a < \min\{1, a_n(\nu - 1)\}. \tag{2.8}$$

Observe that $a_n(n-3) < 1$ and $a_n(n-4) > 1$ for $n = 5, 6, 7, 8$. Hence for $n = 5, 6, 7$ or 8 and $\rho_1 \geq 1$ we have the estimates:

$$(2.9) \qquad \mu_n(a) \le \left\{ \begin{array}{ll} S_n(a, n-2), & a_n(n-2) \le a < a_n(n-3) \\ S_n(a, n-3), & a_n(n-3) \le a < 1 \end{array} \right\} \equiv U_n(a).$$

It follows from (2.3) and (2.5) that

$$\frac{1}{a}\left[(n-1) - (1-a^2)\mu_n(a)\right] \le \text{ Re } \sum_{k=1}^{n-1} \frac{2}{a - z_k} = \text{ Re } \sum_{j=1}^{n-1} \frac{1}{a - \zeta_j} \le \frac{n-1}{\rho_1}.$$

It follows that if $\rho_1 \ge 1$, then $\mu_n(a) \ge \dfrac{n-1}{1+a}$. For $n = 2, 3$ or 4, if we were to assume that $\rho_1 \ge 1$, then by (2.4) we have $\mu_n(a) < \dfrac{n-1}{1+a}$ for $0 < a < 1$, a contradiction. This proves the theorem in these cases. Henceforth we may assume $n = 5, 6, 7$ or 8.

Remark 2.1. For the special case $n = 5$, if we were to assume that $\rho_i \ge 1$ and if we knew that $a_5(3) \le a < 1$, then (2.9) gives $\mu_5(a) < 2.003$. However, since

$$\frac{4}{1+a} \le \mu_5(a) < 2.003,$$

we see that a in fact lies in the smaller interval $A_5' < a < 1$, where

$$A_5' = \frac{4}{2.003} - 1 = 0.997004\cdots.$$

Throughout we let

$$\gamma_j = \frac{\zeta_j - a}{a\zeta_j - 1} \quad \text{ and } \quad w_k = \frac{z_k - a}{az_k - 1}.$$

It is known [5] that if p has the form (2.1), then

$$(2.10) \qquad \prod_{j=1}^{n-1} |\gamma_j| \le \frac{\displaystyle\prod_{k=1}^{n-1} |w_k|}{\displaystyle n - a \sum_{k=1}^{n-1} \text{ Re } w_k}.$$

If in addition p is extremal, then since there is a zero on each closed subarc of $|z| = 1$ of length π, it is known [5] that there exists zeros, say z_{n-1} and z_{n-2}, on $|z| = 1$ such that Re $\{w_{n-1} + w_{n-2}\} \le \dfrac{4a}{1+a^2}$. Hence we also get the estimate

$$(2.11) \qquad \prod_{j=1}^{n-1} |\gamma_j| \le \frac{\displaystyle\prod_{k=1}^{n-1} |w_k|}{\displaystyle n - \frac{4a^2}{1+a^2} - a \sum_{k=1}^{n-3} \text{ Re } w_k}.$$

Finally, it was also shown in [5] that for any $p \in \mathcal{P}_n$ of the form (2.1)

$$\text{If } \ |\gamma_{j_0}| < \frac{1}{1+a-a^2}, \ \text{then } \rho_{j_0} < 1, \tag{2.12}$$

and hence

$$\text{If } \ \prod_{j=1}^{M} |\gamma_j| < \frac{1}{(1+a-a^2)^M}, \quad \text{then} \quad \rho_{j_0} < 1, \ \text{for some } j_0. \tag{2.13}$$

3. Proof of Main Results

Throughout this section we tacitly assume that $n = 5, 6, 7$ or 8 and

$$p(z) = (z-a)\prod_{k=1}^{n-1}(z - z_k), \, 0 < a < 1 \tag{3.1}$$

is extremal : $I(\mathcal{P}_n) = I(p) = I(a) = \rho_1 \leq \rho_j$, for $j = 1, 2, \ldots, n-1$. We will make use of the following results whose proofs are deferred to Section 4:

Lemma 3.1. *If* $[1 - (1 - |p(0)|)^{\frac{1}{n}}] \leq \lambda \leq \sin\frac{\pi}{n}, \frac{a(a-\lambda)}{2\lambda} > 1$ *and* $\rho_j \geq 1$ *for* $j = 1, 2, \ldots, n-1,$ *then there exists a critical point* $\zeta_0 = a + \rho_0 e^{i\theta_0}$ *such that* $\cos\theta_0 > -a$*, i.e.,* $Re\ \zeta_0 > 0$.

Lemma 3.2. *If* $|z_{k_0}| < R_n \equiv 1 - (0.91)^n$*, for some zero* $z_{k_0} \neq a$ *or if* $a < A_n$ *, where* A_n *is the smallest positive root of* $n - \frac{4x^2}{1+x^2} - (n-3)x - (1 + x - x^2)^{n-1} = 0$*, then* $\rho_1 < 1$.

n	A_n	R_n
8	*0.4912...*	*0.5297...*
7	*0.5732...*	*0.4832...*
6	*0.6929...*	*0.4321...*
5	*0.8811...*	*0.3759...*

Remark 3.1. It is important to point out that these bounds for A_n satisfy $A_n > a_n(n-2)$ for $n = 5, 6, 7$ and 8 and hence $\mu_n(a)$ can be estimated using (2.9).

Lemma 3.3. *If* $|z_k| \geq R_n$*, for* $k = 1, 2, \ldots, n-1$*, and* $\mu_n(a) = \sum_{k=1}^{n-1} \frac{1}{r_k^2}$*, then*

$$\prod_{j=1}^{n-1} |\zeta_j| \geq \frac{(1-a^2)|p(0)|}{a(n-1)}\left[\frac{(n-1)^2}{n} - \mu_n(a)\right].$$

Lemma 3.4. *If* $\rho_j \geq 1$*, for* $j = 1, 2, \ldots, n-1$ *then*

$$\prod_{j=1}^{n-1} |\zeta_j| \leq \left(\prod_{j=1}^{n-1} \rho_j\right)\left[\left(\frac{2}{n-1}\right)\left\{\sum_{k=1}^{n-1}\frac{|z_k|^2 - a^2}{r_k^2}\right\} - (1-a^2)\right]^{\frac{n-1}{2}}$$

Lemma 3.5. *If* $|z_k| \geq R_n$, $\rho_j \geq 1$ $(k, j = 1, 2, \ldots, n-1)$ *and*

$$\mu_n(a) \leq U_n(a) < \frac{(n-1)^2}{n},$$

then

$$Q_n(a) \leq \mu_n^*(a) \tag{3.2}$$

where

$$Q_n(a) \equiv \left[\frac{(n-1)U_n(a)}{2U_n(a)-(n-1)}\right]\left[\frac{(n-1)^2}{n} - U_n(a)\right]^{\frac{2}{n-1}} \frac{\left(\frac{n}{n-1}\right)^{\frac{2}{n-1}}}{(1-a^2)^{\frac{n-3}{n-1}}}$$

and

$$\mu_n^*(a) \equiv \frac{(n-1-\nu_n)}{R^{2\left(\frac{n-2}{n-1}\right)}} + \frac{(\nu_n - 1)}{R^{-2\left(\frac{n-2}{n-1}\right)}} + R^{\frac{2(n-2)(n-2\nu_n)}{n-1}} \tag{3.3}$$

with $R = \dfrac{2\sin\frac{\pi}{n}}{1+a}$ *and* $\nu_n = \left[\left[\frac{n-1}{2}\right]\right] =$ *smallest integer* $\geq \frac{n-1}{2}$.

Proof of Theorem 1.1 . We have already shown that $I(a) < 1$ when $n = 2, 3$ or 4 and $0 < a < 1$. Also we have $I(0) < 1$ and, by Lemma A, $I(1) \leq 1$. Let us suppose that $n = 5, 6, 7$, or 8. Without loss of generality we suppose p is extremal and has the form (3.1) with $0 < a < 1$. Assume $\rho_1 \geq 1$. By Lemma 3.2 we must then have $R_n \leq |z_k| \leq 1$, for all $k = 1, 2, \ldots, n-1$, and $A_n \leq a < 1$, where A_n is as given in the table for $n = 5, 6, 7$ and 8. We point out that for the special case $n = 5$, since $A_5 > a_5(3)$, Remark 2.1 allows us to restrict a even further, namely $A_5' < a < 1$. Hence in what follows, we let $A_5 = A_5' = 0.997\cdots$. Now using the estimates for $\mu_n(a)$ given by (2.9) we check that for $n = 5, 6, 7$ or 8,

$$\mu_n(a) \leq U_n(a) < \frac{(n-1)^2}{n}$$

and also that

$$Q_n(a) - \mu_n^*(a) > 0, \quad A_n \leq a < 1. \tag{3.4}$$

We apply Lemma 3.5 , but then (3.2) contradicts (3.4). Hence $\rho_1 < 1$. $\square$

Proof of Corollary 1.1. Since $I(0) < 1$ and the proof of the theorem shows that $I(a) < 1$ when $0 < a < 1$, we see that p cannot be extremal for any $0 \leq a < 1$ and $n = 2, 3, \ldots, 8$. Thus since p is extremal , we must have $a = 1$. Hence by Lemma A, the extremal polynomial has the form $p(z) = z^n - 1$. The other extremal polynomials are just rotations of p. $\square$

4. Proofs of Lemmas

Recall that p has the form (3.1), $I(\mathcal{P}_n) = I(p) = I(a) = \rho_1 \leq \rho_j$, for $j = 1, 2, \ldots, n-1$ and $n = 5, 6, 7$ or 8.

Proof of Lemma 3.1. This proof uses essentially the same idea as in Brown [6]. However, here we make use of a result due to Borcea [3] which generalizes a result of Bojanov, Rahman and Szynal [1]:

Lemma C. *If $p \in \mathcal{P}_n$, $0 < a < 1$ and $\rho_j \geq 1$ for $j = 1, 2, \ldots, n-1$, then $|p(z)| > 1 - (1-\lambda)^n$ for $|z - a| = \lambda \leq \sin\frac{\pi}{n}$.*

Using our hypothesis, we apply Lemma C to conclude that

$$|p(z)| > 1 - (1-\lambda)^n \geq |p(0)|, \quad |z - a| = \lambda.$$

Since p is univalent in $|z-a| \leq \lambda$, it follows that there exists a unique point z_0 with $|z_0 - a| < \lambda$ such that $p(0) = p(z_0)$. Without loss of generality $\operatorname{Im} z_0 \geq 0$ (else consider $\overline{p(\overline{z})}$). By a variant of the Grace-Heawood Theorem (see [1] for example), there exists a critical point in each of the half-planes bounded by the perpendicular bisector Γ_0 of the segment from 0 to z_0. Let $\zeta_0 = a + \rho_0 e^{i\theta_0}$ be the critical point in the half-plane containing z_0. We claim that Γ_0 intersects the imaginary axis at a point ω_0 outside $|z| = 1$ (hence $\operatorname{Re} \zeta_0 > 0$ and so $\cos\theta_0 > -a$). To verify this claim let

$$z^* \equiv \left(\frac{a-\lambda}{a}\right)\left[\sqrt{a^2-\lambda^2} + i\lambda\right].$$

This is the point on the line which is tangent to the circle $|z - \lambda| = a$ and which passes through the origin with $\operatorname{Im} z^* > 0$ and $|z^*| = a - \lambda$. Let Γ^* be the perpendicular bisector of the segment from 0 to z^*. Since $|z^*| = a - \lambda$, it is evident that Γ^* meets the imaginary axis at a point ω^* with $0 < \operatorname{Im} \omega^* \leq \operatorname{Im} \omega_0$. Since $\operatorname{Im} \omega^* = \dfrac{a(a-\lambda)}{2\lambda} > 1$ the claim is proved. $\square$

<u>**Remark 4.1**</u>. One can easily improve the estimate given in Lemma 3.1 as follows. It is simple to check that the perpendicular bisector Γ^* meets the real axis at the point $r^* = \dfrac{a(a-\lambda)}{2\sqrt{a^2-\lambda^2}}$. A brief sketch shows that since $\rho_0 \geq 1$, then $\cos\theta_0 > \mu_0 - a$, where μ_0 satisfies $\dfrac{\mu_0}{|\omega^*| - 1} = \dfrac{r^*}{|\omega^*|}$. Thus, we obtain

$$\cos\theta_0 > \mu_0 - a, \quad \text{where} \quad \mu_0 = \frac{a(a-\lambda) - 2\lambda}{2\sqrt{a^2-\lambda^2}}.$$

It then follows that

$$|\gamma_0| = \left|\frac{\zeta_0 - a}{a\zeta_0 - 1}\right| > \frac{1}{\sqrt{(1-a^2)^2 + a^2 - 2a(1-a^2)(\mu_0 - a)}}.$$

<u>**Proof of Lemma 3.2**</u>. From (2.11) it follows that

$$\prod_{j=1}^{n-1} |\gamma_j| \leq \frac{1}{n - \frac{4a^2}{1+a^2} - (n-3)a} \equiv \phi_n(a).$$

Now since $\phi_n(0) = \frac{1}{n}$, we see that $\phi_n(a) < \dfrac{1}{(1+a-a^2)^{n-1}}$ for $a < A_n$ for some $A_n > 0$. By (2.13), we then have $\rho_{j_0} < 1$ for some j_0. Clearly A_n is the smallest positive root of the equation $n - (n-3)x - \frac{4x^2}{1+x^2} - (1+x-x^2)^{n-1} = 0$.

Suppose now that $|z_{k_0}| < R_n = 1-(0.91)^n$. Assume $\rho_j \geq 1$ for all $j = 1,2,\ldots,n-1$. Thus $A_n \leq a < 1$ by the above. For $n = 5, 6$ or 7, we apply Lemma 3.1 with $\lambda = 0.09$ to conclude that there exists a critical point $\zeta_0 = a + \rho_0 e^{i\theta_0}$ such that $\cos\theta_0 > -a$. It follows that

$$|\gamma_0| = \left|\frac{\zeta_0 - a}{a\zeta_0 - 1}\right| > \frac{1}{\sqrt{1+a^2-a^4}}$$

and since $|z_{k_0}| < R_n$ we have $|w_{k_0}| < \left|\dfrac{a+R_n}{1+aR_n}\right| \equiv B_n$. From (2.10), we conclude that for some γ_{j_0},

$$\frac{|\gamma_{j_0}|^{n-2}}{\sqrt{1+a^2-a^4}} < \prod_{j=1}^{n-1} |\gamma_j| < \frac{B_n}{n-a(n-1)} \equiv \psi_n(a).$$

An easy check shows that

$$|\gamma_{j_0}|^{n-2} < \sqrt{1+a^2-a^4}\ \psi_n(a) < \frac{1}{(1+a-a^2)^{n-2}}, \quad A_n \leq a < 1$$

for $n = 5, 6$ or 7. Hence by (2.12), $\rho_{j_0} < 1$, a contradiction.

Similarly for $n = 8$ and $\lambda = 0.09$, we use Remark 4.1 which yields

$$\frac{|\gamma_{j_0}|^6}{\Delta} < \prod_{j=1}^{7} |\gamma_j| \leq \frac{B_n}{8-7a},$$

where $\Delta = \sqrt{(1-a^2)^2 + a^2 - 2a(1-a^2)(\mu_0 - a)}$. However, $\dfrac{B_n\,\Delta}{8-7a} < \dfrac{1}{(1+a-a^2)^6}$ for $A_8 \leq a < 1$, which again gives $\rho_{j_0} < 1$, a contradiction. $\square$

Proof of Lemma 3.3. If $z_k = xe^{i\theta}$ and $R_n \leq x \leq 1$, then we first assert that

$$\frac{n}{(n-1)a}\left(\frac{|z_k|^2 - a^2}{r_k^2}\right) + \left(\frac{1-|z_k|^2}{|z_k|}\cos\theta\right) \leq \frac{n}{(n-1)a}\left(\frac{1-a^2}{r_k^2}\right). \tag{4.1}$$

If $\cos\theta \leq 0$ or $x = 1$, then (4.1) is true. Suppose $\cos\theta > 0$ and $R_n \leq x < 1$. Let

$$g(x) \equiv (x^2 - a^2) + \frac{(n-1)a}{n}\left(\frac{1-x^2}{x}\right) r_k^2 \cos\theta.$$

It suffices to show $g(x) \leq 1 - a^2$.

Observe that since $r_k^2 = a^2 + x^2 - 2ax\cos\theta$, we have

$$g(x) \leq (x^2 - a^2) + \left(\frac{n-1}{8n}\right)\frac{(1-x^2)(a^2+x^2)^2}{x^2} \equiv G(x).$$

Now $G(x) \leq 1 - a^2$ holds if and only if

$$\phi(x) \equiv x^4 + x^2\left(2a^2 - \frac{8n}{n-1}\right) + a^4 \leq 0.$$

An easy check shows that $\phi(1) < 0$ and $\phi(R_n) < 0$ and hence $\phi(x) < 0$ for $R_n \leq x < 1$ when $n = 5, 6, 7$ or 8. Now since $g(x) \leq G(x) \leq 1 - a^2$, the result (4.1) is proved.

Secondly we assert that if $\rho_1 \geq 1$ then

$$\text{(4.2)} \qquad \operatorname{Re} \sum_{j=1}^{n-1} \zeta_j \leq \frac{1}{a} \ \left[\sigma_n(a) - (n-1)(1-a^2)\right],$$

where $\sigma_n(a) = \displaystyle\sum_{j=1}^{n-1} \frac{|z_k|^2 - a^2}{r_k^2}$.

To see this, observe by (2.5) we have

$$\frac{|w|^2 - a^2}{r^2} = 1 - a \ \operatorname{Re} \left\{\frac{2}{a-w}\right\} \quad \text{for } r = |a-w|$$

and since $\zeta_j = a + \rho_j e^{it_j}$, we get from (2.3)

$$\begin{aligned}\sigma_n(a) &= (n-1) - a \ \operatorname{Re} \sum_{j=1}^{n-1} \frac{1}{a - \zeta_j} \\ &\geq (n-1) + a \sum_{j=1}^{n-1} \cos t_j.\end{aligned}$$

Using this we obtain (4.2):

$$\begin{aligned}\operatorname{Re} \sum_{j=1}^{n-1} \zeta_j &\leq \sum_{j=1}^{n-1} (a + \cos t_j) \\ &\leq (n-1)a + \left\{\frac{\sigma_n(a) - (n-1)}{a}\right\} \\ &= \frac{1}{a} \left[\sigma_n(a) - (n-1)(1-a^2)\right].\end{aligned}$$

Let $z_k = |z_k| e^{i\theta_k}$ and suppose that $\operatorname{Re} z_k > 0$ for $k = 1, \ldots, m$ and all other zeros except a lie in $\operatorname{Re} z \leq 0$. Now we know that

$$\text{(4.3)} \qquad \begin{aligned}\prod_{j=1}^{n-1} |\zeta_j| &= \frac{|p(0)|}{n} \left| \frac{1}{a} + \sum_{k=1}^{n-1} \frac{1}{z_k} \right| \\ &\geq -\frac{|p(0)|}{n} \left\{ \operatorname{Re} \left[\frac{1}{a} + \sum_{k=1}^{n-1} \frac{1}{z_k} \right] \right\}.\end{aligned}$$

Making use of (4.1), (4.2) and the fact that the centers of mass of the zeros and critical points of p are identical, we obtain

$$\begin{aligned}
&\operatorname{Re}\left[\frac{1}{a}+\sum_{k=1}^{n-1}\frac{1}{z_k}\right] = \frac{1-a^2}{a} + \operatorname{Re}\left[a+\sum_{k=1}^{n-1} z_k\right] + \operatorname{Re}\left[\sum_{k=1}^{n-1}\left(\frac{1}{z_k}-\overline{z}_k\right)\right] \\
&\le \frac{1-a^2}{a} + \left(\frac{n}{n-1}\right) \operatorname{Re}\sum_{k=1}^{n-1}\zeta_j + \sum_{k=1}^{m}\frac{1-|z_k|^2}{|z_k|}\cos\theta_k \\
&\le \frac{1-a^2}{a} + \left(\frac{n}{(n-1)a}\right)\left[\sum_{k=1}^{n-1}\frac{|z_k|^2-a^2}{r_k^2} - (n-1)(1-a^2)\right] + \sum_{k=1}^{m}\frac{1-|z_k|^2}{|z_k|}\cos\theta_k \\
&= -(n-1)\left(\frac{1-a^2}{a}\right) + \sum_{k=1}^{m}\left\{\frac{n}{(n-1)a}\frac{|z_k|^2-a^2}{r_k^2} + \frac{1-|z_k|^2}{|z_k|}\cos\theta_k\right\} \\
&\qquad + \frac{n}{(n-1)a}\sum_{k=m+1}^{n-1}\frac{|z_k|^2-a^2}{r_k^2} \\
&\le -(n-1)\left(\frac{1-a^2}{a}\right) + \sum_{k=1}^{m}\left\{\frac{n}{(n-1)a}\frac{(1-a^2)}{r_k^2}\right\} + \frac{n}{(n-1)a}(1-a^2)\sum_{k=m+1}^{n-1}\frac{1}{r_k^2} \\
&= \left(\frac{1-a^2}{a}\right)\left[\left(\frac{n}{n-1}\right)\mu_n(a) - (n-1)\right].
\end{aligned}$$

This inequality and (4.3) give the desired estimate:

$$\prod_{j=1}^{n-1}|\zeta_j| \ge \frac{(1-a^2)|p(0)|}{a(n-1)}\left[\frac{(n-1)^2}{n} - \mu_n(a)\right].$$

(If there are no zeros in $\operatorname{Re} z > 0$ other than a, then the estimate still holds.) $\square$

Proof of Lemma 3.4. Apply the identity (2.5) to (2.3) to get

$$\sum_{j=1}^{n-1}\frac{a^2-|\zeta_j|^2}{\rho_j^2} = (n-1) + 2\sum_{k=1}^{n-1}\frac{a^2-|z_k|^2}{r_k^2},$$

and since $\rho_j \ge 1$, for $j = 1, 2, \ldots, n-1$,

$$\sum_{j=1}^{n-1}\frac{|\zeta_j|^2}{\rho_j^2} \le 2\sum_{k=1}^{n-1}\frac{|z_k|^2-a^2}{r_k^2} - (n-1)(1-a^2).$$

Apply the arithmetic-geometric means inequality:

$$\begin{aligned}
\prod_{j=1}^{n-1}\frac{|\zeta_j|}{\rho_j} &= \left(\prod_{j=1}^{n-1}\frac{|\zeta_j|^2}{\rho_j^2}\right)^{\frac{1}{2}} \le \left[\left(\frac{1}{n-1}\sum_{j=1}^{n-1}\frac{|\zeta_j|^2}{\rho_j^2}\right)^{n-1}\right]^{1/2} \\
&\le \left[\frac{2}{n-1}\sum_{k=1}^{n-1}\frac{|z_k|^2-a^2}{r_k^2} - (1-a^2)\right]^{\frac{n-1}{2}}.
\end{aligned}$$

$\square$

Before embarking on the proof of the last lemma, we first prove:

Proposition 4.1. *If* $\sigma_0' = \sum_{\substack{k=1\\k\neq k_0}}^{n-1} \frac{t_k^2 - a^2}{r_k^2}$, $R_n \leq |z_k| \leq t_k \leq 1$ *and* $m = \frac{2}{n-1}$, *then*

$$\frac{m\left(\frac{x^2-a^2}{r_{k_0}^2} + \sigma_0'\right) - (1-a^2)}{x^m} \leq m\left(\frac{1-a^2}{r_{k_0}^2} + \sigma_0'\right) - (1-a^2), \tag{4.4}$$

where $R_n \leq |z_{k_0}| \leq x \leq 1$.

Proof. Without loss of generality $x < 1$. Note that (4.4) holds if and only if

$$(x^2-a^2) + r_{k_0}^2\sigma_0' - \frac{1}{m}(1-a^2)r_{k_0}^2 \leq x^m\left\{\left[1-a^2+r_{k_0}^2\sigma_0'\right] - \frac{r_{k_0}^2}{m}(1-a^2)\right\}$$

and this holds if and only if

$$(x^2-x^m) + (1-x^m)\left[-a^2 + r_{k_0}^2\left\{\sigma_0' - \frac{1}{m}(1-a^2)\right\}\right] \leq 0,$$

if and only if

$$\frac{(x^2-x^m)}{(1-x^m)} - a^2 + r_{k_0}^2\left\{\sigma_0' - \frac{1}{m}(1-a^2)\right\} \leq 0 \tag{*}$$

Observe first that

$$\begin{aligned} r_{k_0}^2\left\{\sigma_0' - \frac{1}{m}(1-a^2)\right\} &\leq r_{k_0}^2\left[(1-a^2)\sum_{k=1}^{n-1}\frac{1}{r_k^2} - \frac{(1-a^2)}{r_{k_0}^2} - \frac{1}{m}(1-a^2)\right] \\ &< (1-a^2)r_{k_0}^2\left\{\mu_n(a) - \frac{1}{m}\right\} + a^2 - x^2 \end{aligned}$$

It follows that (*) holds if

$$\psi_n(x) \equiv \frac{x^m(x^2-1)}{1-x^m} + r_{k_0}^2(1-a^2)\left\{\mu_n(a) - \frac{1}{m}\right\} \leq 0 \tag{**}$$

Let $H(x) = \frac{x^m(x^2-1)}{1-x^m}$ and observe that H decreases with x and is negative. Using the estimates for $\mu_n(a)$ given in (2.9) for $n = 5, 6, 7$ or 8, we check that

$$\psi_n(x) \leq H(R_n) + (1+a)^2(1-a^2)\left(U_n(a) - \frac{1}{m}\right) < 0, \quad A_n \leq a < 1.$$

Thus inequality (**) and hence the proposition are true. □

Proof of Lemma 3.5. We apply Lemmas 3.3 and 3.4 with $m = \frac{2}{n-1}$ to get:

$$\frac{(1-a^2)|p(0)|}{a(n-1)}\left[\frac{(n-1)^2}{n} - \mu_n(a)\right] \le \left(\prod_{j=1}^{n-1}\rho_j\right)\left[m\sum_{k=1}^{n-1}\frac{|z_k|^2-a^2}{r_k^2} - (1-a^2)\right]^{\frac{n-1}{2}}$$

or,

$$\frac{(1-a^2)}{(n-1)}\left[\frac{(n-1)^2}{n} - \mu_n(a)\right] \le \left(\prod_{j=1}^{n-1}\rho_j\right)\Phi_n(a)^{\frac{n-1}{2}} \tag{4.5}$$

where

$$\Phi_n(a) \equiv \left[\frac{m\sum_{k=1}^{n-1}\frac{|z_k|^2-a^2}{r_k^2} - (1-a^2)}{|z_1|^m|z_2|^m\cdots|z_{n-1}|^m}\right].$$

Using Proposition 4.1 first with $x = |z_1|$ and $t_k = |z_k|$ for $k = 2, 3, \ldots, n-1$, we obtain

$$\frac{m\left[\frac{|z_1|^2-a^2}{r_1^2} + \sum_{\substack{k=1\\k\neq 1}}^{n-1}\frac{|z_k|^2-a^2}{r_k^2}\right] - (1-a^2)}{|z_1|^m|z_2|^m\cdots|z_{n-1}|^m} \le \frac{m\left[\frac{1-a^2}{r_1^2} + \sum_{\substack{k=1\\k\neq 1}}^{n-1}\frac{t_k^2-a^2}{r_k^2}\right] - (1-a^2)}{t_2^m t_3^m\cdots t_{n-1}^m}$$

Now let $x = t_2 = |z_2|$, $t_1 = 1$ and $t_k = |z_k|$ for $k = 3, 4, \ldots, n-1$ and apply Proposition 4.1 to the right-hand side to get

$$\Phi_n(a) \le \frac{m\left[\frac{1-a^2}{r_2^2} + \sum_{\substack{k=1\\k\neq 2}}^{n-1}\frac{t_k^2-a^2}{r_k^2}\right] - (1-a^2)}{t_3^m t_4^m\cdots t_{n-1}^m}$$

Next, we let $x = t_3 = |z_3|$, $t_1 = t_2 = 1$ and $t_k = |z_k|$ for $k = 4, \ldots, n-1$. After applying Proposition 4.1 $n-1$ times we conclude that

$$\Phi_n(a) \le (1-a^2)\left[m\mu_n(a) - 1\right].$$

(Since $\rho_1 \ge 1$, we already pointed out that $\mu_n(a) \ge \dfrac{n-1}{1+a} > \dfrac{1}{m}$.) Hence (4.5) then yields

$$\frac{\left(\frac{n}{n-1}\right)\left[\frac{(n-1)^2}{n} - \mu_n(a)\right]}{(1-a^2)^{\frac{n-3}{2}}} \le \left(n\prod_{j=1}^{n-1}\rho_j\right)\left[\frac{2}{n-1}\sum_{k=1}^{n-1}\frac{1}{r_k^2} - 1\right]^{\frac{n-1}{2}}. \tag{4.6}$$

The next step is to estimate the right-hand side of (4.6). To do this we note that

$$\left(n\prod_{j=1}^{n-1}\rho_j\right)^{\frac{2}{n-1}}\sum_{k=1}^{n-1}\frac{1}{r_k^2}=\sum_{k=1}^{n-1}\frac{1}{R_k^2}, \tag{4.7}$$

where

$$R_k=\frac{r_k}{\left(\prod_{k=1}^{n-1}r_k\right)^{\frac{1}{n-1}}}\quad \text{for}\quad k=1,2,\ldots,n-1.$$

Note also that since $\rho_1\geq 1$, the estimate (2.4) gives $2\sin\frac{\pi}{n}\leq r_k\leq 1+a$ and hence

$$R^{\frac{n-2}{n-1}}\leq R_k\leq R^{-\left(\frac{n-2}{n-1}\right)},$$

where $R=\dfrac{2\sin\frac{\pi}{n}}{1+a}$. Clearly $\prod_{k=1}^{n-1}R_k=1$. Using Lemma B, we choose the smallest integer ν so that

$$\left(R^{\frac{n-2}{n-1}}\right)^{n-1-\nu}\left(R^{-\left(\frac{n-2}{n-1}\right)}\right)^{\nu}=R^{\frac{(n-2)(n-1-2\nu)}{(n-1)}}\geq 1. \tag{4.8}$$

If $n\geq 5$ and $a>2\sin\frac{\pi}{5}-1=0.1755...$, we see that $R<1$ and hence (4.8) holds when $\nu\geq\frac{n-1}{2}$. Let $\nu_n=$ the smallest integer $\geq\frac{n-1}{2}$. From Lemma B we then conclude that

$$\sum_{k=1}^{n-1}\frac{1}{R_k^2}\leq\mu_n^*(a), \tag{4.9}$$

where $\mu_n^*(a)$ is defined by (3.3).

Using (4.9) and (4.7) in (4.6) we see that

$$\frac{\left(\frac{n}{n-1}\right)\left[\frac{(n-1)^2}{n}-\mu_n(a)\right]}{(1-a^2)^{\frac{n-3}{2}}}\leq\left[\frac{2}{n-1}\mu_n^*(a)-\left(n\prod_{j=1}^{n-1}\rho_j\right)^{\frac{2}{n-1}}\right]^{\frac{n-1}{2}}. \tag{4.10}$$

On the other hand (4.6) also yields

$$\frac{\left(\dfrac{n}{n-1}\right)^{\frac{2}{n-1}}\left[\dfrac{(n-1)^2}{n}-\mu_n(a)\right]^{\frac{2}{n-1}}}{(1-a^2)^{\frac{n-3}{n-1}}\left[\dfrac{2}{n-1}\mu_n(a)-1\right]}\leq\left(n\prod_{j=1}^{n-1}\rho_j\right)^{\frac{2}{n-1}}. \tag{4.11}$$

Using this inequality in (4.10) we have

$$\frac{\left(\dfrac{n}{n-1}\right)^{\frac{2}{n-1}}\left[\dfrac{(n-1)^2}{n}-\mu_n(a)\right]^{\frac{2}{n-1}}}{(1-a^2)^{\frac{n-3}{n-1}}}\left[\frac{(n-1)\mu_n(a)}{2\mu_n(a)-(n-1)}\right]\leq\mu_n^*(a).$$

The result now follows by observing that $\mu_n/(2\mu_n - n + 1)$ is a decreasing function of μ_n. $\square$

5. Remarks

This technique is useful in studying the Sendov Conjecture but cannot as yet provide a proof for arbitrary n. The principal drawback to this technique is the requirement that $\sum_{k=1}^{n-1} \frac{1}{r_k^2} < \frac{(n-1)^2}{n}$. We can however use the technique to prove the conjecture for polynomials of arbitrary degree n, but with at most eight distinct zeros:

Theorem 5.1. *If $p(z) = \prod_{k=1}^{8} (z - z_k)^{m_k} \in \mathcal{P}_n, \sum_{k=1}^{8} m_k = n$, then each of the disks $|z - z_k| \le 1$ for $k = 1, 2, \ldots, n$ contains a critical point of p.*

Proof. Let $\mathcal{P}_n(8) \subset \mathcal{P}_n$ denote the class of all polynomials in $\mathcal{P}_n$ with at most eight distinct zeros. It was shown in [5] that there still exists an extremal polynomial $p \in \mathcal{P}_n(8)$ with $I(\mathcal{P}_n(8)) = I(p) = I(a)$. (By a rotation, we assume that $0 \le a \le 1$.) If $a = 0$, $a = 1$ or a is not a simple zero then $I(a) \le 1$ and we are done. Hence we assume p is extremal and has the form

$$p(z) = (z - a) \prod_{k=1}^{7} (z - z_k)^{m_k}, \quad 0 < a < 1 \text{ with } \sum_{k=1}^{7} m_k = n - 1. \tag{5.1}$$

Note also that

$$p'(z) = \left(n \prod_{j=1}^{7} (z - \zeta_j) \right) \left(\prod_{k=1}^{7} (z - z_k)^{m_k - 1} \right). \tag{5.2}$$

Because of (5.2), we see that (2.10) and (2.11) give

$$\begin{aligned} \left(\prod_{j=1}^{7} |\gamma_j| \right) \left(\prod_{k=1}^{7} |w_k|^{m_k - 1} \right) &\le \frac{\prod_{k=1}^{7} |w_k|^{m_k}}{n - a \sum_{k=1}^{7} m_k \text{ Re } w_k} \\ &\le \frac{\prod_{k=1}^{7} |w_k|^{m_k}}{n - \frac{4a^2}{1+a^2} - (n-3)a}. \end{aligned} \tag{5.3}$$

It follows that

$$\prod_{k=1}^{7} |\gamma_j| \le \frac{1}{n - (n-1)a} \le \frac{1}{n - \frac{4a^2}{1+a^2} - (n-3)a}. \tag{5.4}$$

If $n \geq 11$, then by (5.4)

$$\prod_{j=1}^{7} |\gamma_j| \leq \frac{1}{n-(n-1)a} \leq \frac{1}{(1+a-a^2)^7}$$

for all $0 < a < 1$, and by (2.13) we get $\rho_{j_0} \leq 1$, so we are done. In view of Theorem 1.1, there are only two cases remaining: $n = 9$ and $n = 10$. Note that

$$\prod_{j=1}^{7} |\gamma_j| \leq \frac{1}{n - \frac{4a^2}{1+a^2} - (n-3)a} \leq \frac{1}{(1+a-a^2)^7}$$

for $n = 9$ if $a > 0.918$ or $a < 0.562$; and for $n = 10$ if $a > 0.8$ or $a < 0.68$. For the remainder of this proof we assume by way of contradiction that $\rho_j \geq 1$ for $j = 1, 2, \ldots, n-1$. Hence by (2.13) when $n = 9$, we must have $0.562 \leq a \leq 0.918$; while for $n = 10$, we have $0.68 \leq a \leq 0.8$.
$\underline{n = 10}$ **:** In this case we first assert that the extremal polynomial can only have two possible forms :

(5.5) $p(z) = (z-a)(z-z_0)^3(z-z_6)Q(z)$ or $p(z) = (z-a)(z-z_0)^2(z-z_6)^2Q(z)$,

where $Q(z) = \prod_{k=1}^{5}(z - z_k)$ and all the z_k are distinct. To see this, suppose p has neither of these forms, then since $n = 10$ and there are at most eight distinct zeros, p' would have three of its nine zeros in common with p and from (5.3) we can now cancel *three* common terms to obtain

$$\prod_{j=1}^{6} |\gamma_j| \leq \frac{1}{10-9a}.$$

Now since $\dfrac{1}{10-9a} < \dfrac{1}{(1+a-a^2)^6}$ for $0.68 \leq a \leq 0.8$, it follows that $\rho_{j_0} < 1$ for some j_0. Contradiction. Hence p has one of the forms (5.5).

Next, there are at most two zeros $z_k \neq a$ in $\operatorname{Re} z \geq 0$. If there are three or more, then for each such z_k, we know that $\operatorname{Re} w_k \leq \dfrac{2a}{1+a^2}$. Hence (5.3) yields

$$\prod_{j=1}^{7} |\gamma_j| \leq \frac{1}{10 - a\left(\dfrac{6a}{1+a^2} + 6\right)}. \tag{5.6}$$

Now note that since $\rho_j \geq 1$, we have $|\gamma_j| \geq \dfrac{\rho_j}{1-a^2+a\rho_j} \geq \dfrac{1}{1+a-a^2}$ and so

$$\frac{1}{(1+a-a^2)^6}\frac{\rho_{j_1}}{(1-a^2+a\rho_{j_1})} \leq \prod_{j=1}^{7} |\gamma_j| \leq \frac{1}{10 - a\left(\dfrac{6a}{1+a^2} + 6\right)}.$$

Hence

$$\rho_{j_1} \le \frac{(1-a^2)}{(h(a)-a)} < 1, \quad \text{for} \quad 0.68 \le a \le 0.8,$$

where $h(a) = \dfrac{10 - a\left(\dfrac{6a}{1+a^2}+6\right)}{(1+a-a^2)^6}$. Contradiction. Thus Re $z \ge 0$ contains at most two zeros $z_k \ne a$.

<u>*Case 1 :*</u> *There is a repeated zero in Re $z \ge 0$.* If z_{k_0} is repeated and Re $z_{k_0} \ge 0$, then by the above and (5.5) this is the only zero in this region other that a. By the extremality of p, we must have $|z_{k_0}| = 1$ and we pointed out earlier that there must exist another zero , say z_1, such that $|z_1| = 1$ and Re $(w_{k_0} + w_1) \le \dfrac{4a^2}{1+a^2}$. However, w_{k_0} is repeated and so we obtain inequality (5.6) again and by the above, $\rho_{j_1} < 1$, a contradiction.

<u>*Case 2 :*</u> *No repeated zeros in Re $z \ge 0$.* In this case, we let $\lambda = 1-(1-a)^{\frac{1}{10}}$ and observe that $\dfrac{a(a-\lambda)}{2\lambda} > 1$ for $0.68 \le a \le 0.8$ and hence Lemma 3.1 gives the existence of a critical point ζ_0 with Re $\zeta_0 > 0$ and hence $|\gamma_0| > \dfrac{1}{\sqrt{1+a^2-a^4}}$. Now since there are no repeated zeros in Re $z \ge 0$, we obtain from (5.4) that for some γ_{j_0},

$$\frac{|\gamma_{j_0}|^6}{\sqrt{1+a^2-a^4}} < \prod_{j=1}^{7} |\gamma_j| \le \frac{1}{10-9a}.$$

Since $\dfrac{\sqrt{1+a^2-a^4}}{10-9a} < \dfrac{1}{(1+a-a^2)^6}$, for $0.68 \le a \le 0.8$, we get by (2.12) that $\rho_{j_0} < 1$, a contradiction.

<u>$n = 9$</u> **:** Here we want to be able to apply Lemma 3.5 and get a contradiction to (3.2) as in the proof of Theorem 1.1. . Tracing back, we must verify several preliminary results.

We set $R_9 = \frac{1}{2}$ and $A_9 = 0.562$. Since p is extremal and $\rho_j \ge 1$, we assert that it must have the form

$$p(z) = (z-a)(z-z_0)^2 \prod_{k=1}^{6} (z-z_k), \quad z_k \text{ distinct.} \tag{5.7}$$

If not, then p and p' have two zeros in common and (5.3) gives

$$\prod_{j=1}^{6} |\gamma_j| \le \frac{1}{9-8a} < \frac{1}{(1+a-a^2)^6}, \; 0.562 \le a \le 0.918$$

and so $\rho_{j_0} < 1$, for some j_0, a contradiction. Thus p has only one repeated zero.

If we let $\lambda = 1-(1-a)^{\frac{1}{9}}$, then $\dfrac{a(a-\lambda)}{2\lambda} > 1$ for $0.562 \le a \le 0.918$ and so by Lemma 3.1 and Remark 4.1, there exists a critical point $\zeta_0 = a + \rho_0 e^{i\theta_0}$ such that

$\cos\theta_0 > \mu_0 - a$, where $\mu_0 = \dfrac{a(a-\lambda)}{2\sqrt{a^2-\lambda^2}}$. If the repeated zero z_0 satisfies $\operatorname{Re} z_0 < 0$, then $z_0 \neq \zeta_0$ and so from (5.4), for some j_0

$$\frac{|\gamma_{j_0}|^6}{\Delta} < \prod_{j=1}^{7} |\gamma_j| \leq \frac{1}{9 - a\left(\dfrac{4a}{1+a^2} + 6\right)},$$

where $\Delta = \sqrt{(1-a^2)^2 + a^2 - 2a(1-a^2)(\mu_0 - a)}$. It is easy to check that

$$\frac{\Delta}{9 - a\left(\dfrac{4a}{1+a^2} + 6\right)} \leq \frac{1}{(1+a-a^2)^6}$$

for $0.562 \leq a \leq 0.918$. Hence $\rho_{j_0} < 1$, a contradiction. Thus we must have $\operatorname{Re} z_0 \geq 0$.

We also need the estimate

$$\mu_9(a) = \sum_{k=1}^{8} \frac{1}{r_k^2} \leq 5.95 \quad (r_7 = r_8 = |a - z_0|). \tag{5.8}$$

To verify this we set $r = |a - z_0|$ and note that since z_0 is repeated ($r = r_7 = r_8 = \rho_8$), $\operatorname{Re} z_0 \geq 0$ and $\rho_j \geq 1$, we must have

$$1 \leq r \leq \sqrt{1+a^2}$$

$$R \equiv 2\sin\frac{\pi}{9} \leq r_k \leq 1 + a \quad \text{for} \quad k = 1, 2, \ldots, 6$$

and by (2.2)

$$\prod_{k=1}^{6} r_k \geq \frac{9}{\sqrt{1+a^2}} \equiv C.$$

Let

$$\nu = \left[\left[\frac{\log\left(\frac{C}{R^6}\right)}{\log\left(\frac{1+a}{R}\right)}\right]\right]$$

and observe that $\nu \geq 5$. From Lemma B we conclude that

$$\mu_9(a) = \sum_{k=1}^{8} \frac{1}{r_k^2} \leq 2 + \left\{\frac{(6-\nu)}{R^2} + \frac{(\nu-1)}{(1+a)^2} + \left[\frac{R^{6-\nu}(1+a)^{\nu-1}}{C}\right]^2\right\} \equiv B(a,\nu).$$

Clearly if $a < 0.7$, then $B(a,6) < B(a,5)$; while if $a \geq 0.7$, then $\nu = 5$. It follows that $\mu_9(a) \leq B(a,5) \leq B(0.562,5) \leq 5.95$ for $0.562 \leq a \leq 0.918$, and this verifies (5.8).

Lemma 3.3 will hold if inequality (4.1) holds and thus it suffices to show that $x^4 + x^2(2a^2 - 9) + a^4 \leq 0$ for $A_9 \leq a < 1$ and $R_9 \leq x < 1$. This is clearly true. Lemma 3.4 holds in any case. Lemma 3.5 can be applied if

$$\psi_9(x) \leq H(0.5) + (1+a)^2(1-a^2)(U_9(a) - 4) < 0, \quad A_9 \leq a \leq 0.918.$$

If we let $U_9(a) \equiv 5.95$, then this holds. Thus we are now in a position to apply Lemma 3.5. Using $U_9(a) = 5.95$, we compute as in the proof of Theorem 1.1 that $Q_9(a) - \mu_9^*(a) > 0$ for $A_9 \leq a \leq 0.918$, contradicting (3.2). This completes the proof of the theorem. $\square$

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References

1. B. Bojanov, Q.I. Rahman, and J. Szynal, *On a conjecture of Sendov about the critical points of a polynomial*, Math. Z. **190** (1985), 281–285.
2. I. Borcea, *On the Sendov Conjecture for polynomials with at most six distinct zeros*, J. Math. Anal. and Appl. **200** (1996), 182-206.
3. I. Borcea, *The Sendov Conjecture for polynomials with at most seven distinct zeros*, Analysis **16** (1996), 137-159.
4. D. A. Brannan, *On a conjecture of Ilieff*, Proc. Cambridge Philos. Soc. **64** (1968), 83–85.
5. J. E. Brown, *On the Ilieff-Sendov Conjecture*, Pacific J. Math **135** (1988), 223-232.
6. J. E. Brown, *On the Sendov Conjecture for sixth degree polynomials*, Proc. Amer. Math. Soc. **113** (1991), 939–946.
7. J. E. Brown, *A proof of the Sendov Conjecture for polynomials of degree seven*, Complex Variables Theory and Appl **33** (1997), 75-95.
8. W. K. Hayman, *Research problems in function theory*, Athlone Press, London (1967).
9. E. S. Katsoprinakis, *Erratum To 'On the Sendov-Ilyeff Conjecture'*, Bull. London Math. Soc. **28** (1996), 605-612.
10. M. Marden, *Geometry of polynomials*, Amer. Math. Soc. Surveys, no. 3 (1966).
11. M. Marden, *Conjectures on the critical points of a polynomial*, Amer. Math. Monthly **90** (1980), 267–276.
12. A. Meir and A. Sharma, *On Ilyeff's Conjecture*, Pacific J. Math. **31** (1969), 459–467.
13. D. Phelps and R. Rodriguez, *Some properties of extremal polynomials for the Ilieff conjecture*, Kodai Math. Sem. Report **24** (1972), 172–175.
14. Z. Rubinstein, *On a problem of Ilyeff*, Pacific J. Math **26** (1968), 159–161.
15. G. Schmeisser, *On Ilieff's Conjecture*, Math. Z. **156** (1977), 165–173.

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