

Robin Hartshorne's Algebraic Geometry Solutions

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APPENDIX C; THE WEIL CONJECTURES

Exercise 5.1. Let $X = \coprod_i X_i$. Obviously, then, $N_r(X) = \sum_i N_r(X_i)$ so that

$$\begin{aligned} Z(X, t) &= \exp\left(\sum_{r=1}^{\infty} N_r(X) \frac{t^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty} \sum_i N_r(X_i) \frac{t^r}{r}\right) \\ &= \exp\left(\sum_i \sum_{r=1}^{\infty} N_r(X_i) \frac{t^r}{r}\right) = \prod_i \exp\left(\sum_{r=1}^{\infty} N_r(X_i) \frac{t^r}{r}\right) = \prod_i Z(X_i, t). \end{aligned}$$

Exercise 5.2. The point is to compute the number of $k_r = \mathbb{F}_{q^r}$ -rational points, i.e. to compute N_r in $\bar{X} = \mathbb{P}_{\bar{k}}^n$. We can consider the following *stratification* of \mathbb{P}^n :

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1} = \dots = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \dots \cup \mathbb{A}^1 \cup \{*\}.$$

Hence for a field k_r of q^r elements, $\mathbb{P}_{k_r}^n$ has $N_r = 1 + q^r + q^{2r} + \dots + q^{nr}$ points. Hence,

$$\begin{aligned} Z(\mathbb{P}^n, t) &= \exp\left(\sum_{r=1}^{\infty} \{1 + q^r + \dots + q^{nr}\} \frac{t^r}{r}\right) = \prod_{i=0}^n \exp\left(\sum_{r=1}^{\infty} \frac{(q^i t)^r}{r}\right) \\ &= \prod_{i=0}^n \exp(-\log(1 - q^i t)) = \prod_{i=0}^n \frac{1}{1 - q^i t} = \frac{1}{(1-t)(1-qt) \dots (1-q^n t)}. \end{aligned}$$

Obviously, this is a rational function so that (1.1) is true. Also, we have $P_{\text{odd}}(t) = 1$, $P_{2i}(t) = 1 - q^i t$ so that (1.3) is true, because $|q^i| = q^{\frac{2i}{2}}$, indeed.

Let's find the self intersection number E of Δ in $\mathbb{P}^n \times \mathbb{P}^n$. Note that $CH^*(\mathbb{P}^n \times \mathbb{P}^n) \simeq CH^*(\mathbb{P}^n) \otimes CH^*(\mathbb{P}^n)$ so that in particular, we have

$$CH^n(\mathbb{P}^n \times \mathbb{P}^n) \simeq \bigoplus_{i+j=n, 0 \leq i, j \leq n} \mathbb{Z} s^i t^j,$$

where $s^i t^j$ corresponds to an n -cycle $\mathbb{P}^i \times \mathbb{P}^j$ in $\mathbb{P}^n \times \mathbb{P}^n$. Hence if $\Delta = \sum_{i,j} a_{ij} s^i t^j$, then if we look at the intersection product, $(\Delta, (\mathbb{P}^i \times \mathbb{P}^j)) = 1$. Also, $(s^i t^j) \cdot (s^{i'} t^{j'}) = 1$ iff $i + i' = n$ and $j + j' = n$ and otherwise it is 0. Hence each $a_{ij} = 1$. That is,

$$E = (\Delta, \Delta) = \left(\sum_{i,j} s^i t^j\right) \left(\sum_{i',j'} s^{i'} t^{j'}\right) = n + 1.$$

Now,

$$\begin{aligned} Z(\mathbb{P}^n, \frac{1}{q^n t}) &= \frac{1}{(1 - \frac{1}{q^n t})(1 - \frac{1}{q^{n-1} t}) \dots (1 - \frac{1}{t})} = \frac{(q^n t)(q^{n-1} t) \dots (t)}{(q^n t - 1)(q^{n-1} t - 1) \dots (t - 1)} \\ &= (-1)^{n+1} \frac{q^{\frac{n(n+1)}{2}} t^{n+1}}{(1-t)(1-qt) \dots (1-q^n t)} = (-1)^{n+1} q^{\frac{nE}{2}} t^E Z(\mathbb{P}^n, t), \end{aligned}$$

so that we have (1.2).

Now, obviously, from $Z(\mathbb{P}^n, t)$, we see that $B_i = 0$ if i is odd and $B_i = 1$ if i is even. Hence $E = n + 1 = \sum_{i=0}^{2n} (-1)^i B_i$, indeed. Also, for $\mathbb{P}_{\mathbb{C}}^n$, we had $H^i(\mathbb{P}_{\mathbb{C}}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i: \text{even} \\ 0 & i: \text{odd} \end{cases}$, so that B_i indeed is $\text{rk} H^i(\mathbb{P}_{\mathbb{C}}^n, \mathbb{Z})$. This one shows (1.4). Hence \mathbb{P}^n satisfies the Weil conjectures.

Exercise 5.3. Obviously, $N_r(X \times \mathbb{A}^1) = q^r N_r(X)$. Hence,

$$\begin{aligned} Z(X \times \mathbb{A}^1, t) &= \exp\left(\sum_{r=1}^{\infty} N_r(X \times \mathbb{A}^1) \frac{t^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty} q^r N_r(X) \frac{t^r}{r}\right) \\ &= \exp\left(\sum_{r=1}^{\infty} N_r(X) \frac{(qt)^r}{r}\right) = Z(X, qt). \end{aligned}$$

Exercise 5.4. Let $N_r(x)$ be the contribution of the closed point $x \in X$ to N_r in X . A \mathbb{F}_{q^r} -rational point is determined by the number of morphisms $\text{Spec} \mathbb{F}_{q^r} \rightarrow X$, which is the same as to give a \mathbb{F}_q -homomorphism $k(x) \rightarrow \mathbb{F}_{q^r}$ and $N_r(x)$ is the number of all such embeddings. This is possible iff $\deg x = [k(x) : \mathbb{F}_q] | r$, and, the number of all such embeddings is just $\deg x$. (If you are confused, consider, say $\deg x = 1$. How many \mathbb{F}_q -linear maps are there? Just 1!.)

Thus, in fact, $N(x) = q^{\deg x}$ so that

$$\begin{aligned} \zeta_X(x) &= \prod_{x \in X} \frac{1}{1 - N(x)^{-s}} = \prod_{x \in X} \frac{1}{1 - (q^{\deg x})^{-s}} = \prod_{x \in X} \frac{1}{1 - (q^{-s})^{\deg x}} = \exp\left(\sum_{x \in X} \sum_{r=1}^{\infty} \frac{(q^{-s} \deg x)^r}{r}\right) \\ &= \exp\left(\sum_{x \in X} \sum_{r=1}^{\infty} \frac{(q^{-s})^r \deg x}{r}\right) = \exp\left(\sum_{x \in X} \sum_{r=1}^{\infty} \frac{(\deg x)(q^{-s})^r \deg x}{r \deg x}\right) = \exp\left(\sum_{x \in X} \sum_{n=1}^{\infty} \frac{N_n(x)(q^{-s})^n}{n}\right) \\ &= \exp\left(\sum_{n=1}^{\infty} N_n \frac{(q^{-s})^n}{n}\right) = Z(X, q^{-s}). \end{aligned}$$

Exercise 5.5. By the Weil conjectures, since $\dim H^1(X, \mathbb{Q}_l) = B_1 = \deg P_1(t) = 2g$, for some α_i , we have

$$\begin{aligned} Z(X, t) &= \frac{P_1(t)}{(1-t)(1-qt)} = \frac{\prod_{i=1}^{2g} (1 - \alpha_i t)}{(1-t)(1-qt)} \\ &= \exp\left(\sum_{r=1}^{\infty} \left(\frac{t^r}{r} + \frac{q^r t^r}{r} - \sum_{i=1}^{2g} \alpha_i^r \frac{t^r}{r}\right)\right) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right). \end{aligned}$$

Hence, $N_r = 1 + q^r - \sum_{i=1}^{2g} \alpha_i^r$ for all $r \geq 1$.

Now, from the functional equation, we have

$$Z\left(\frac{1}{qt}\right) = \pm q^{1-g} t^{2-2g} Z(t).$$

The left hand side of the equation is,

$$\frac{\prod_{i=1}^{2g} (1 - \frac{\alpha_i}{qt})}{(1 - \frac{1}{qt})(1 - \frac{1}{t})} = \frac{t^{2-2g} q^{1-g} \prod_{i=1}^{2g} (\sqrt{qt} - \frac{\alpha_i}{\sqrt{q}})}{(1-t)(1-qt)}.$$

Hence, by comparing terms, we have

$$P_1(t) = \prod_{i=1}^{2g} (1 - \alpha_i t) = \pm \prod_{i=1}^{2g} \left(\sqrt{qt} - \frac{\alpha_i}{\sqrt{q}}\right).$$

Recall that $\frac{1}{-a+bt} = \frac{-1}{a} (1 + \frac{b}{a}t + (\frac{b}{a})^2 t^2 + \dots)$ so that $\log(-a+bt) = -(\frac{b}{a}t + (\frac{b}{a})^2 t^2 + \dots)$. Hence, if we replace $P_1(t)$ be the right hand side of above,

$$Z(X, t) = \exp\left(\sum_{r=1}^{\infty} \left(\frac{t^r}{r} + \frac{q^r t^r}{r} + \sum_{i=1}^{2g} \frac{q^r t^r}{\alpha_i^r r}\right)\right) = \exp\left(\sum_{r=1}^{\infty} \left(1 + q^r + q^r \sum_{i=1}^{2g} \frac{1}{\alpha_i^r} \frac{t^r}{r}\right)\right).$$

Hence, $N_r = 1 + q^r + q^r \sum_{i=1}^{2g} \frac{1}{\alpha_i^r}$ as well, which was also $1 + q^r - \sum_{i=1}^{2g} \alpha_i^r$. Since we know N_1, N_2, \dots, N_g , we hence know all of $\sum_{i=1}^{2g} \alpha_i^r$ for $-g \leq r \leq g$. Using some combinatorial argument and Newton's identity on symmetric polynomials, above information is enough to determine all $\sum \alpha_i^r$ for $r > g$ as well. Hence $N_r = 1 + q^r - \sum \alpha_i^r$ is determined as well.

Exercise 5.6. From IV, Exercise 4.16, $N_r = q^r - (f^r + \check{f}^r) + 1$. Hence,

$$\begin{aligned} Z(t) &= \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty} (q^r - (f^r + \check{f}^r) + 1) \frac{t^r}{r}\right) \\ &= \frac{1}{1-qt} \frac{1}{1-t} (1-ft)(1-\check{f}t) = \frac{1-at+qt^2}{(1-t)(1-qt)}, \end{aligned}$$

since $f\check{f} = q$ and $a = f + \check{f} \in \mathbb{Z}$.

Now from the functional equation, we will have

$$P_1(t) = (1-ft)(1-\check{f}t) = \pm(\sqrt{q}t - \frac{f}{\sqrt{q}})(\sqrt{q}t - \frac{\check{f}}{\sqrt{q}})$$

so that

$$|a| = |f + \check{f}| \leq 2g \Leftrightarrow |f| = |\check{f}| = \sqrt{q}.$$

(See Exercise 5.7, (b) and (c).)

Exercise 5.7.

(a). We have

$$\begin{aligned} Z(X, t) &= \frac{P_1(t)}{(1-t)(1-qt)} = \frac{\prod_{i=1}^{2g} (1 - \alpha_i t)}{(1-t)(1-qt)} \\ &= \exp\left(\sum_{r=1}^{\infty} \frac{(1 - \sum_{i=1}^{2g} \alpha_i^r + q^r) t^r}{r}\right). \end{aligned}$$

Hence, $N_r = 1 - \sum_{i=1}^{2g} \alpha_i^r + q^r = 1 - a_r + q^r$ so that $a_r = \sum_{i=1}^{2g} (\alpha_i)^r$.

(b). (\Leftarrow) If $|\alpha_i| \leq \sqrt{q}$ for all i , then,

$$|a_i| \leq \sum_{i=1}^{2g} |\alpha_i|^r \leq \sum_{i=1}^{2g} \sqrt{q^r} = 2g\sqrt{q^r}.$$

(\Rightarrow) Consider the following easy power series expansion:

$$\sum_{i=1}^{2g} \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{r=1}^{\infty} a_r t^r.$$

Since $|a_r| \leq 2g\sqrt{q^r}$, the RHS is holomorphic in $|t| < \frac{1}{\sqrt{q}}$. If $|\alpha_i| > \sqrt{q}$ for some i , then, the LHS has a pole of order 1 in $|t| < \frac{1}{\sqrt{q}}$ at $t = \frac{1}{\alpha_i}$ which is hence a contradiction.

(c). By using the functional equations, as in Ex (5.5), we have

$$P_1(t) = \prod_{i=1}^{2g} (1 - \alpha_i t) = \pm \prod_{i=1}^{2g} (\sqrt{q}t - \frac{\alpha_i}{\sqrt{q}}).$$

Hence,

$$\{\alpha_1^{-1}, \dots, \alpha_{2g}^{-1}\} = \{\frac{\alpha_1}{q}, \dots, \frac{\alpha_{2g}}{q}\}$$

so that for all j , there is a unique number $i(j)$ with $\alpha_j^{-1} = \frac{\alpha_{i(j)}}{q}$, so that $|\alpha_i| \leq \sqrt{q}$ then implies $|\alpha_j^{-1}| = \frac{|\alpha_i|}{q} \leq \frac{1}{\sqrt{q}}$ i.e. $|\alpha_j| \geq \sqrt{q}$ for all j . This proves that $|\alpha_i| = \sqrt{q}$.