**Travels of a floating space tool.**

A spacecraft is in a circular orbit 6750 km from the Earth's center of mass (i.e, it is about 250 miles above the surface). Two astronauts are doing a repair on the outside. One of them throws a 1 kg tool to the other, at a speed of 10 meters/sec, in a direction exactly toward the center of the Earth. The second astronaut is unable to reach it as it flies by.

**1.** (a) Using data from table 13.3 on p. 942 of the text, calculate the orbital period of the spacecraft, in minutes. (Ignore all forces except the earth's gravitational pull. Note that the earth's mass $M$ is given only to four significant digits, so any $=$ sign resulting from a calculation involving $M$ in what follows is limited to that degree of accuracy.)

(b) Say why the speed of the spacecraft is constant, and calculate it, in meters/sec.

Solution. (a) From the table, the Earth's mass $M = 5.975 \cdot 10^{24}$ kg, and the gravitational constant is $G = 6.6726$ Newtons·m$^2$·kg$^{-2}$.

The oft-appearing product $MG = 3.9869 \cdot 10^{14}$ Newtons·m$^2$·kg$^{-1}$.

Kepler's third law gives the period as

$$\sqrt{\frac{4\pi^2(6.75 \times 10^6)^3}{3.9869 \cdot 10^{14}}} = 5518 \text{ sec} = 91.97 \text{ min}.$$

(b) Choose polar coordinates with origin at the Earth's center of mass. At time $t$ the speed along a circle of radius $r$ is $r\dot{\theta}$, which is constant because $r$ is constant and because, by Kepler's second law (or conservation of angular momentum), $r^2\dot{\theta}$ is constant. [Another valid justification suggested during class is that the acceleration is directed toward the Earth's center, perpendicular to the velocity, so it affects direction of motion but not speed.] By part (a), $\dot{\theta} = 2\pi/5518$, so the speed is

$$2\pi(6.75 \times 10^6)/5518 = 7686 \text{ meters/sec } (= 17190 \text{ miles/hr}).$$

**2.** Calculate the angular momentum of the tool about the Earth's center.

Solution. Choose coordinates as in the solution of 1(b),

For a second 1 kg tool in the astronaut's pocket, the angular momentum would be the same, because initially the two tools have the same position vector, and the initial difference in the velocity vectors lies along the position vector, so disappears when you take cross product with the position vector. (Or just note that they have the same $\dot{\theta}$ initially because the difference in their velocities lies along a radius of the circular orbit.) So the angular momentum is

$$A = r^2\dot{\theta} = (6.75 \times 10^6)^2 \times 2\pi/5518 = 5.188 \times 10^{10}.$$

**3.** Show that the subsequent orbit of the tool is an ellipse. Calculate the eccentricity and semimajor axis.

Solution. As shown in class, the eccentricity of the tool's orbit is

$$e = \sqrt{1 + \frac{2EA^2}{mM^2G^2}},$$

where $E$ is the total energy, potential + kinetic. This is given by

$$E = \frac{1}{2}\left(\dot{r}^2 + \frac{A^2}{r^2} - \frac{2MG}{r}\right) = \frac{1}{2}\left(100 + \frac{5.188^2 \times 10^{20}}{6.75^2 \times 10^{12}}\right) - \frac{3.9869 \times 10^{14}}{6.75 \times 10^6}$$
$$= -2.9528 \times 10^7.$$

But that's useless, because the energy $E_0$ of the second tool differs from $E$ only by 50; and this difference doesn't show up when we use only 4 decimal places, as above, in the specification of $E$ (and we can't reliably use more). So if we continue, we'll just get the original circular orbit.

What we can do, however, is write $E = E_0 + 50$, and note that since the orbit of the second tool is circular—so of eccentricity 0—therefore

$$1 + \frac{2E_0A^2}{mM^2G^2} = 0.$$

Then we get that

$$e = \sqrt{\frac{100A^2}{M^2G^2}} = \frac{10A}{MG} = \frac{5.188 \times 10^{11}}{3.98688 \times 10^{14}} = 1.3 \times 10^{-3} = .0013.$$

Since $e < 1$, the orbit is an ellipse (nearly a circle). For a suitable choice of the coordinate $\theta$ (so that it's 0 when the distance of the tool from the origin is minimal), the equation of the ellipse is

$$r = \frac{A^2/MG}{1 + e\cos\theta}.$$

Applying this to the second tool (which has the same $A$), we get $6.75 \times 10^6 = A^2/MG$.

[A calculation with the above values of $A$, $M$, and $G$ would have given

$$A = (5.188 \times 10^{10})^2/(3.9869 \times 10^{14}) = 6.7512 \times 10^6.$$

This is a bit less accurate because of the roundoff error which accompanies multiplication and division.] So, the orbit of the tool is given by

$$r = \frac{6.75 \times 10^6}{1 + .0013\cos\theta}$$

Taking $\theta = 0$ and $\theta = \pi$ in this equation, we find that the major axis has half-length

$$6.75 \times 10^6/(1 - .0013^2) = 6.75(1 + .0013^2 + \textit{negligeable}) \times 10^6 = 6.750011.4 \times 10^6.$$

Thus the semimajor axis is about 11.4 meters longer than the radius of the circular orbit.

**4.** You might think at first that since the tool was originally thrown toward the Earth's center, it should always be closer to the Earth than the spacecraft. Is this true?

Solution. Apparently not, since there are times when it is 11.4 farther out!

The negative answer could have been deduced, at least qualitatively, right off, because the orbits of the tool and the spacecraft cross, and since the spacecraft orbit is circular, the orbit of the tool must be an ellipse, close to the circle since the velocities at time t=0 differ by very little, and so with small eccentricity, but with major axis bigger than the radius of the circle.

**5.** What is the approximate distance between the spacecraft and the tool when the tool first returns to the point at which it was thrown?

Solution. From Kepler's third law we compute

$$\frac{dT}{da} = 4.72 \cdot 10^{-7}\sqrt{a}.$$

Setting $a = 6750000$ and $\Delta a = 11.4$ we get the approximation $\Delta T = .014$. So the tool first arrives back at the initial point about .014 seconds after the spacecraft, during which time the spacecraft has traveled $7686 \times .014 \approx 107$ meters.

**6.** What would happen if you stood in the space station with your feet toward the earth and pushed a floating pen gently toward the floor, so gently that it should take an hour or so to reach the floor.

Solution. The pen would behave something like the above tool. So after half an orbit or so, it would start moving back up (at least relative to your initial position, though now you'd be upside down), and very briefly reach a position higher than its initial one, and a small bit closer to you.