Math 182 Recitation3-20

Due at recitation, Thurs. Mar. 20, 2008

1. (a) Let $C = \mathbf{r}(t) = \big(g(t), h(t)\big),\ a \le t \le b$, be a plane curve. Let $\tau = \psi(t)$, where ψ is a strictly increasing function, with inverse $\phi = \psi^{-1}$.

Then the plane curve $\tilde{C} = \tilde{\mathbf{r}}(\tau) = \big(g(\phi(\tau)), h(\phi(\tau))\big),\ \psi(a) \le \tau \le \psi(b)$, has the same underlying path as C, with the same initial and terminal points. Use the change of variable (substitution) formula for one-variable integrals to show that for any vector field F,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\tilde{C}} \mathbf{F} \cdot d\,\tilde{\mathbf{r}}.$$

(b) In (a), suppose that ψ is strictly decreasing, so that while $\tilde{C}$ still has the same underlying path as C, the initial and terminal points get reversed. In other words, now $\psi(b) \le \tau \le \psi(a)$, and the path is traversed in the opposite direction. Show that in this case,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{\tilde{C}} \mathbf{F} \cdot d\,\tilde{\mathbf{r}}.$$

Note. The simplest case of what happens in (b) is the formula $\int_a^b f(t)\,dt = -\int_b^a f(\tau)\,d\tau$ that you get by the substitution $t = -\tau$.

2. In this problem you will fill in the omitted part of the proof of Theorem 1 in §16.3, by showing that if path independence holds for a vector field

$$\mathbf{F} = \big(M(x,y,z), N(x,y,z), P(x,y,z)\big)$$

in a connected open region R, then $\mathbf{F} = \nabla f$ for some function f defined on R.

Fix a point (a,b,c) in R. For (x,y,z) in R, define $f(x,y,z)$ to be $\int_C \mathbf{F} \cdot d\mathbf{r}$ along any curve $C = \mathbf{r}(t)$ in R from (a,b,c) to (x,y,z). (By assumption, such curves exist—since R is connected, and they all produce the same value for $f(x,y,z)$.)

(a) Explain why $f(x',y',z') - f(x,y,z)$ is $\int_{C'} \mathbf{F} \cdot d\mathbf{r}'$ along any curve $C' = \mathbf{r}'(t)$ in R from (x,y,z) to (x',y',z').

(b) Show that the partial derivative f_x is M by using part (a) to express the difference $f(x+u,y,z) - f(x,y,z)$ for small u as an integral along the straight line segment joining (x,y,z) to $(x+u,y,z)$, and then differentiating that expression with respect to u. (Note that R is assumed open, and so there is a ball centered at (x,y,z) and contained entirely within R; and therefore the said segment will lie entirely within R if u is sufficiently small.)

A similar argument, which you needn't repeat, shows that $f_y = N$ and $f_z = P$.