# PROBLEM SET 2

## Conditional Probability and Independence

1. Let $A$, $B$, $C$ and $D$ be events such that $B = A'$, $C \cap D = \emptyset$, and
$P[A] = \frac{1}{4}$, $P[B] = \frac{3}{4}$, $P[C|A] = \frac{1}{2}$, $P[C|B] = \frac{3}{4}$, $P[D|A] = \frac{1}{4}$, $P[D|B] = \frac{1}{8}$
Calculate $P[C \cup D]$.
A) $\frac{5}{32}$ B) $\frac{1}{4}$ C) $\frac{27}{32}$ D) $\frac{3}{4}$ E) 1

2. You are given that $P[A] = .5$ and $P[A \cup B] = .7$.
Actuary 1 assumes that $A$ and $B$ are independent and calculates $P[B]$ based on that assumption. Actuary 2 assumes that $A$ and $B$ mutually exclusive and calculates $P[B]$ based on that assumption. Find the absolute difference between the two calculations.
A) 0 B) .05 C) .10 D) .15 E) .20

3. (SOA) An actuary studying the insurance preferences of automobile owners makes the following conclusions:
(i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.
(ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15 .
What is the probability that an automobile owner purchases neither collision nor disability coverage?
A) 0.18 B) 0.33 C) 0.48 D) 0.67 E) 0.82

4. Two bowls each contain 5 black and 5 white balls. A ball is chosen at random from bowl 1 and put into bowl 2. A ball is then chosen at random from bowl 2 and put into bowl 1. Find the probability that bowl 1 still has 5 black and 5 white balls.
A) $\frac{2}{3}$ B) $\frac{3}{5}$ C) $\frac{6}{11}$ D) $\frac{1}{2}$ E) $\frac{6}{13}$

 

5. (SOA) An insurance company examines its pool of auto insurance customers and gathers the following information:
(i) All customers insure at least one car.
(ii) 70% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.
Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.
A) 0.13 B) 0.21 C) 0.24 D) 0.25 E) 0.30

6. (SOA) An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims. Calculate the probability that a claim submitted to the insurance company includes operating room charges.
A) 0.10 B) 0.20 C) 0.25 D) 0.40 E) 0.80

7. Let $A$, $B$ and $C$ be events such that $P[A|C] = .05$ and $P[B|C] = .05$. Which of the following statements must be true?
A) $P[A \cap B|C] = (.05)^2$ B) $P[A' \cap B'|C] \geq .90$ C) $P[A \cup B|C] \leq .05$
D) $P[A \cup B|C'] \geq 1 - (.05)^2$ E) $P[A \cup B|C'] \geq .10$

8. A system has two components placed in series so that the system fails if either of the two components fails. The second component is twice as likely to fail as the first. If the two components operate independently, and if the probability that the entire system fails is .28, find the probability that the first component fails.
A) $\frac{.28}{3}$ B) .10 C) $\frac{.56}{3}$ D) .20 E) $\sqrt{.14}$

9. A ball is drawn at random from a box containing 10 balls numbered sequentially from 1 to 10. Let $X$ be the number of the ball selected, let $R$ be the event that $X$ is an even number, let $S$ be the event that $X \geq 6$, and let $T$ be the event that $X \leq 4$. Which of the pairs $(R,S)$, $(R,T)$, and $(S,T)$ are independent?

A) $(R,S)$ only B) $(R,T)$ only C) $(S,T)$ only

D) $(R,S)$ and $(R,T)$ only E) $(R,S)$, $(R,T)$ and $(S,T)$

10. (SOA) A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

A) 0.20 B) 0.25 C) 0.35 D) 0.42 E) 0.57

11. If $E_1$, $E_2$ and $E_3$ are events such that $P[E_1|E_2] = P[E_2|E_3] = P[E_3|E_1] = p$, $P[E_1 \cap E_2] = P[E_1 \cap E_3] = P[E_2 \cap E_3] = r$, and $P[E_1 \cap E_2 \cap E_3] = s$, find the probability that at least one of the three events occurs.

A) $1 - \frac{r^3}{p^3}$ B) $\frac{3p}{r} - r + s$ C) $\frac{3r}{p} - 3r + s$ D) $\frac{3p}{r} - 6r + s$ E) $\frac{3r}{p} - r + s$

12. (SOA) A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease. Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

A) 0.115 B) 0.173 C) 0.224 D) 0.327 E) 0.514

16. (SOA) An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policy-holder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year. A policyholder dies in the next year. What is the probability that the deceased policyholder was ultra-preferred?

A) 0.0001 B) 0.0010 C) 0.0071 D) 0.0141 E) 0.2817

17. (SOA) The probability that a randomly chosen male has a circulation problem is 0.25 . Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem. What is the conditional probability that a male has a circulation problem, given that he is a smoker?

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{2}{5}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

18. (SOA) A study of automobile accidents produced the following data:

| Model year | Proportion of all vehicles | Probability of involvement in an accident |
|---|---|---|
| 1997 | 0.16 | 0.05 |
| 1998 | 0.18 | 0.02 |
| 1999 | 0.20 | 0.03 |
| Other | 0.46 | 0.04 |

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997 .

A) 0.22 B) 0.30 C) 0.33 D) 0.45 E) 0.50

# PROBLEM SET 2 SOLUTIONS

1. Since $C$ and $D$ have empty intersection, $P[C \cup D] = P[C] + P[D]$.
Also, since $A$ and $B$ are "exhaustive" events (since they are complementary events, their union is the entire sample space, with a combined probability of
$P[A \cup B] = P[A] + P[B] = 1$ ).
We use the rule $P[C] = P[C \cap A] + P[C \cap A']$, and the rule $P[C|A] = \frac{P[A \cap C]}{P[A]}$ to get
$P[C] = P[C|A] \cdot P[A] + P[C|A'] \cdot P[A'] = \frac{1}{2} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{11}{16}$ and
$P[D] = P[D|A] \cdot P[A] + P[D|A'] \cdot P[A'] = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{3}{4} = \frac{5}{32}$.
Then, $P[C \cup D] = P[C] + P[D] = \frac{27}{32}$. Answer: C.

2. Actuary 1: Since $A$ and $B$ are independent, so are $A'$ and $B'$.
$P[A' \cap B'] = 1 - P[A \cup B] = .3$.
But $.3 = P[A' \cap B'] = P[A'] \cdot P[B'] = (.5)P[B'] \rightarrow P[B'] = .6 \rightarrow P[B] = .4$.
Actuary 2: $.7 = P[A \cup B] = P[A] + P[B] = .5 + P[B] \rightarrow P[B] = .2$.
Absolute difference is $|.4 - .2| = .2$. Answer: E

3. We identify the following events:
$D =$ an automobile owner purchases disability coverage, and
$C =$ an automobile owner purchases collision coverage.
We are given that
(i) $P[C] = 2P[D]$, (ii) $C$ and $D$ are independent, and (iii) $P[C \cap D] = .15$.
From (ii) it follows that $P[C \cap D] = P[C] \cdot P[D]$, and therefore,
$.15 = 2P[D] \cdot P[D] = 2(P[D])^2$, from which we get $P[D] = \sqrt{.075} = .27386$.
Then, $P[C] = 2P[D] = .54772$, $P[D'] = 1 - P[D] = .72614$, and
$P[C'] = 1 - P[C] = .45228$.
Since $C$ and $D$ are independent, so are $C'$ and $D'$, and therefore, the probability that an automobile owner purchases neither disability coverage nor collision coverage is
$P[C' \cap D'] = P[C'] \cdot P[D'] = .328$. Answer: B

4. Let $C$ be the event that bowl 1 has 5 black balls after the exchange.
Let $B_1$ be the event that the ball chosen from bowl 1 is black, and
let $B_2$ be the event that the ball chosen from bowl 2 is black.
Event $C$ is the disjoint union of $B_1 \cap B_2$ and $B_1' \cap B_2'$ (black-black or white-white picks), so that $P[C] = P[B_1 \cap B_2] + P[B_1' \cap B_2']$ .
The black-black combination has probability $(\frac{6}{11})(\frac{1}{2})$ ,
since there is a $\frac{5}{10}$ chance of picking black from bowl 1, and then (with 6 black in bowl 2, which now has 11 balls) $\frac{6}{11}$ is the probability of picking black from bowl 2. This is
$P[B_1 \cap B_2] = P[B_2|B_1] \cdot P[B_1] = (\frac{6}{11})(\frac{1}{2})$ .
In a similar way, the white-white combination has probability $(\frac{6}{11})(\frac{1}{2})$ .
Then $P[C] = (\frac{6}{11})(\frac{1}{2}) + (\frac{6}{11})(\frac{1}{2}) = \frac{6}{11}$ . Answer: C

5. We identify the following events:
$A$ - the policyholder insures exactly one car (so that $A'$ is the event that the policyholder insures more than one car), and
$S$ - the policyholder insures a sports car.
We are given $P[A'] = .7$ (from which it follows that $P[A] = .3$) , and $P[S] = .2$ (and $P[S'] = .8$). We are also given the conditional probability $P[S|A'] = .15$ ;
"of those customers who insure more than one car", means that we are looking at a conditional event given $A'$ .
We are asked to find $P[A \cap S']$ .
We create the following probability table, with the numerals in parentheses indicating the order in which calculations are performed.

| | $A$, .3 | $A'$, .7 |
|---|---|---|
| $S$, .2 | **(2)** $P[S \cap A]$ $= P[S] - P[S \cap A']$ $= .2 - .105 = .095$ | **(1)** $P[S \cap A'] = P[S\|A'] \cdot P[A']$ $= (.15)(.7) = .105$ |
| $S'$, .8 | **(3)** $P[A \cap S']$ $= P[A] - P[A \cap S]$ $= .3 - .095 = .205$ | |

We can solve this problem with a model population of 1000 individuals with auto insurance. $\#A = 300$ (since 70% insure more than one car), and $\#S = 200$. From $P[S|A'] = .15$ we get $\#S \cap A' = .15 \times \#A' = .15 \times 700 = 105$ . Then $\#S \cap A = \#S - \#S \cap A' = 200 - 105 = 95$, and $\#S' \cap A = \#A - \#S \cap A = 300 - 95 = 205$ is the number that insure exactly one car and the car is not a sports car. Therefore $P[S' \cap A] = .205$ . Answer: B

6. We define the following events.

$E$ - the claim includes emergency room charges ,

$O$ - the claim includes operating room charges.

We are given $P[E \cup O] = .85$ , $P[E'] = .25$ and $E$ and $O$ are independent.

We are asked to find $P[O]$ .

We use the probability rule $P[E \cup O] = P[E] + P[O] - P[E \cap O]$ .

Since $E$ and $O$ are independent, we have $P[E \cap O] = P[E] \cdot P[O] = (.75)P[O]$

(since $P[E] = 1 - P[E'] = 1 - .25 = .75$).

Therefore, $.85 = P[E \cup O] = .75 + P[O] - .75P[O]$ .

Solving for $P[O]$ results in $P[O] = .40$ . Answer: D

7. $P[A' \cap B'|C] = P[(A \cup B)'|C] = 1 - P[A \cup B|C] \geq .9$,

since $P[A \cup B|C] \leq P[A|C] + P[B|C] = .1$ . Answer: B

8. $.28 = P[C_1 \cup C_2] = P[C_1] + P[C_2] - P[C_1 \cap C_2] = P[C_1] + 2P[C_1] - 2(P[C_1])^2$

Solving the quadratic equation results in $P[C_1] = .1$ (or 1.4, but we disregard this solution since $P[C_1]$ must be $\leq 1$). Alternatively, each of the five answers can be substituted into the expression above for $P[C_1]$ to see which one satisfies the equation. Answer: B

9. $P[R] = .5$ , $P[S] = .5$ , $P[T] = .4$ .

$P[R \cap S] = P[6,8,10] = .3 \neq (.5)(.5) = P[R] \cdot P[S] \rightarrow R, S$ are not independent

$P[R \cap T] = P[2,4] = .2 = (.5)(.4) = P[R] \cdot P[T] \rightarrow R, T$ are independent

$P[S \cap T] = P[\emptyset] = 0 \neq (.5)(.4) = P[S] \cdot P[T] \rightarrow S, T$ are not independent. Answer: B

 

10. We identify the following events

$N$ - non-smoker , $L$ - light smoker , $H$ - heavy smoker ,

$D$ - dies during the 5-year study .

We are given $P[N] = .50$ , $P[L] = .30$ , $P[H] = .20$ .

We are also told that $P[D|L] = 2P[D|N] = \frac{1}{2}P[D|H]$

(the probability that a light smoker dies during the 5-year study period is $P[D|L]$ ;
it is the conditional probability of dying during the period given that the individual is a light smoker). We wish to find the conditional probability $P[H|D]$ .

We will find this probability from the basic definition of conditional probability,
$P[H|D] = \frac{P[H \cap D]}{P[D]}$ . These probabilities can be found from the following probability table.
The numerals indicate the order in which the calculations are made.

We are not given specific values for $P[D|L]$ , $P[D|N]$ , or $P[D|H]$ , so will let $P[D|N] = k$ , and then $P[D|L] = 2k$ and $P[D|H] = 4k$ .

| | $N$ , .5 | $L$ , .3 | $H$ , .2 |
|---|---|---|---|
| $D$ | **(1)** $P[D \cap N]$ $= P[D|N] \cdot P[N]$ $= (k)(.5) = .5k$ | **(2)** $P[D \cap L]$ $= P[D|L] \cdot P[L]$ $= (2k)(.3) = .6k$ | **(3)** $P[D \cap H]$ $= P[D|H] \cdot P[H]$ $= (4k)(.2) = .8k$ |

**(4)** $P[D] = P[D \cap N] + P[D \cap L] + P[D \cap H] = .5k + .6k + .8k = 1.9k$ .

**(5)** $P[H|D] = \frac{P[H \cap D]}{P[D]} = \frac{.8k}{1.9k} = .42$ . Answer: D

11. $P[E_1|E_2] = \frac{P[E_1 \cap E_2]}{P[E_2]} = p \rightarrow P[E_2] = \frac{r}{p}$ , and similarly $P[E_3] = P[E_1] = \frac{r}{p}$ .

Then, $P[E_1 \cup E_2 \cup E_3]$

$= P[E_1] + P[E_2] + P[E_3] - (P[E_1 \cap E_2] + P[E_1 \cap E_3] + P[E_2 \cap E_3])$

$+ P[E_1 \cap E_2 \cap E_3] = 3(\frac{r}{p}) - 3r + s$ . Answer: C

12. In this group of 937 man, we regard proportions of people with certain conditions to be probabilities. We are given the population of 937 men. We identify the following conditions:
$DH$ - died from causes related to heart disease , and
$PH$ - had a parent with heart disease.
We are given $\#PH = 312$ , so if follows that $\#PH' = 937 - 312 = 625$.
We are also given $\#DH = 210$ and $\#DH \cap PH = 102$ .
It follows that $\#DH \cap (PH') = \#DH - \#DH \cap PH = 210 - 102 = 108$ .
Then the probability of dying due to heart disease given that neither parent suffered from heart disease is the proportion $\frac{\#DH\cap(PH')}{\#PH'} = \frac{108}{625}$ .

The solution in terms of conditional probability rules is as follows. From the given information, we have
$P[DH] = \frac{210}{937}$ (proportion who died from causes related to heart disease)
$P[PH] = \frac{312}{937}$ (proportion who have parent with heart disease)
$P[DH|PH] = \frac{102}{312}$ (prop. who died from heart disease given that a parent has heart disease).
We are asked to find $P[DH|PH']$ ($PH'$ is the complement of event $PH$, so that $PH'$ is the event that neither parent had heart disease). Using event algebra, we have
$P[DH|PH] = \frac{P[DH\cap PH]}{P[PH]} \Rightarrow P[DH \cap PH] = P[DH|PH] \cdot P[PH] = (\frac{102}{312})(\frac{312}{937}) = \frac{102}{937}$ .
We now use the rule $P[A] = P[A \cap B] + P[A \cap \overline{B}]$ .
Then $P[DH] = P[DH \cap PH] + P[DH \cap PH'] \rightarrow \frac{210}{937} = \frac{102}{937} + P[DH \cap PH']$
$\Rightarrow P[DH \cap PH'] = \frac{108}{937}$ .
Finally, $P[DH|PH'] = \frac{P[DH\cap PH']}{P[PH']} = \frac{108/937}{1-P[PH]} = \frac{108/937}{1-\frac{312}{937}} = \frac{108}{625} = .1728$ .

These calculations can be summarized in the following table.

| | | |
|---|---|---|
| | $DH$ , 210<br>given | $DH'$ , 727<br>$= 937 - 210$ |
| $PH$ , 312<br>given | $DH \cap PH = 102$<br>given | $DH' \cap PH = 210$<br>$= 312 - 102$ |
| $PH'$ , 625<br>$= 937 - 312$ | $DH \cap PH' = 108$<br>$= 210 - 102$ | $DH' \cap PH' = 517$<br>$= 727 - 210$ or<br>$= 625 - 108$ |

$P[DH|PH'] = \frac{P[DH\cap PH']}{P[PH']} = \frac{\#[DH\cap PH']}{\#[PH']} = \frac{108}{625} = .1728$ .

In this example, probability of an event is regarded as the proportion of a group that experiences that event. Answer: B

16. continued

From the calculations already made it is easy to find the probability that the deceased policyholder was preferred;

$$P[P|D] = \frac{P[P\cap D]}{P[D]} = \frac{P[D|P]\cdot P[P]}{P[D|S]\cdot P[S]+P[D|P]\cdot P[P]+P[D|U]\cdot P[U]}$$
$$= \frac{(.005)(.4)}{(.01)(.5)+(.005)(.4)+(.001)(.1)} = \frac{.0020}{.0071} = .2817\ .$$

And $P[S|D]$ is $\frac{(.01)(.5)}{(.01)(.5)+(.005)(.4)+(.001)(.1)} = \frac{.0050}{.0071} = .7042$ .

The calculations can be summarized in the following table.

| | $S$ , .5<br>given | $P$ , .4<br>given | $U$ , .1<br>given |
|---|---|---|---|
| $D$ | $P(D\|S) = .01$<br>given<br>$P(D\cap S)$<br>$= P(D\|S)\cdot P(S)$<br>$= (.01)(.5) = .005$ | $P(D\|P) = .005$<br>given<br>$P(D\cap P)$<br>$= P(D\|P)\cdot P(P)$<br>$= (.005)(.4) = .002$ | $P(D\|U) = .001$<br>given<br>$P(D\cap U)$<br>$= P(D\|U)\cdot P(U)$<br>$= (.001)(.1) = .0001$ |

$P(D) = P[D\cap S] + P[D\cap P] + P[D\cap U] = .005 + .002 + .0001 = .0071$ .

$P[U|D] = \frac{P[U\cap D]}{P[D]} = \frac{.0001}{.0071} = .0141$ . Answer: D

17. We identify the following events:

$C$ - a randomly chosen male has a circulation problem ,

$S$ - a randomly chosen male is a smoker.

We are given the following probabilities:

$P[C] = .25$ , $P[S|C] = 2P[S|C']$ .

From the rule $P[A\cap B] = P[A|B]\cdot P[B]$ , we get

$P[S\cap C] = P[S|C]\cdot P[C] = (.25)P[S|C]$ , and

$P[S\cap C'] = P[S|C']\cdot P[C'] = P[S|C']\cdot(1-P[C]) = (.75)(\frac{1}{2})P[S|C]$ ,

so that $P[S] = P[S\cap C] + P[S\cap C'] = (.25)P[S|C] + (.75)(\frac{1}{2})P[S|C] = .625P[S|C]$ .

We are asked to find $P[C|S]$ . This is $P[C|S] = \frac{P[C\cap S]}{P[S]} = \frac{(.25)P[S|C]}{.625P[S|C]} = .4$ .

Note that the way in which information was provided allowed us to formulate various probabilities in terms of $P[S|C]$ (but we do not have enough to find $P[S|C]$). Answer: C