Depth zero base change for $U(3)$

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► Joint with J. Lansky

Goal. To understand base change explicitly via types.

For $U(3)$, base change exists by Rogawski.

We will look at base change for depth zero representations of unramified $U(3)$.

► I. Notation and main result

F: non-archimedean local field, residue characteristic $\neq 2$ and characteristic $\neq 3$ (perhaps even characteristic 0, depending on how much the existence of base change is knwon).

Let E/F be the unramified quadratic extension, k_E, k_F: residue fields. Let E^1, k_E^1 be the norm-1 groups. Let $\Gamma = \mathrm{Gal}(E/F) = \mathrm{Gal}(k_E/k_F) = \{1, \epsilon\}$.

Let $\mathbb{G} = U(3; E/F)$, and $\tilde{\mathbb{G}} = \mathrm{Res}_{E/F}\,\mathbb{G}$, $G = G(F)$, $\tilde{G} = \mathbb{G}(F) \simeq \mathrm{GL}_3(E)$. Let Z and $\tilde{Z}$ be the center of G and $\tilde{G}$. Let $\tilde{\mathcal{B}}$ be the building of $\tilde{G}$ and $\mathcal{B} = \tilde{\mathcal{B}}^\epsilon$ be the building of G.

Let y be a hyperspecial vertex in $\mathcal{B}$ and z a non-hyperspecial vertex. For all $x \in \mathcal{B}$, let G_x be the corresponding parahoric subgroup with reductive quotient M_x. Similarly we have the notations $\tilde{G}_x$, $\tilde{M}_x$.

A depth zero type is a pair $(G_x, \inf(\sigma))$, where σ is an irreducible cuspidal representation of M_x.

Recall: base change exists for finite classical groups (generalization of Shintani liftings).

"Theorem." Let π be an irreducible representation of G containing $(G_x, \inf(\sigma))$. Let Π be the L-packet containing π. Then $BC(\Pi)$ contains the pair $(\tilde{G}_x, \inf(BC(\sigma)))$.

Remark. Notice that $BC(\Pi)$ is a singleton (and is identified with its unique element).

► II. Principal series

Let $\tilde{B} \subset \tilde{G}$ be an ϵ-invariant Borel subgroup, and $B = \tilde{B}^\epsilon$. Then Levi of $B \simeq E^\times \times E^1$. Let $\psi = (\psi_1, \psi_2)$ be a character of $E^\times \times E^1$.

Lemma. Let $\pi = \mathrm{Ind}_B^G \psi$, $\tilde{\pi} = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tilde{\psi}$, where $\tilde{\psi} = \psi \circ (\mathcal{N} : \tilde{G} \to G; g \mapsto g.\epsilon(g))$. Then

$$\Theta_{\tilde{\pi},\epsilon}(g) = \Theta_\pi(\mathcal{N}(g)),$$

for all $g \in \tilde{G}$ such that $\mathcal{N}(g)$ is regular.

Then we are done with irreducible principal series.
Now π reducible $\implies$ ψ has one of 3 forms:

(1) $\psi_1 = |\cdot|^{\pm 1}$. Then the components of π are characters θ of G and $St_G \otimes \theta$, and their base changes are $\theta \circ \mathcal{N}$ or $St_{\tilde{G}} \otimes (\theta \circ \mathcal{N})$.

(2) $\psi_1 \neq \mathbf{1}$, $\psi_1|F^\times = \mathbf{1}$. Then the components of π form an L-packet Π, and $BC(\Pi) = \tilde{\pi}$ (irreducible).

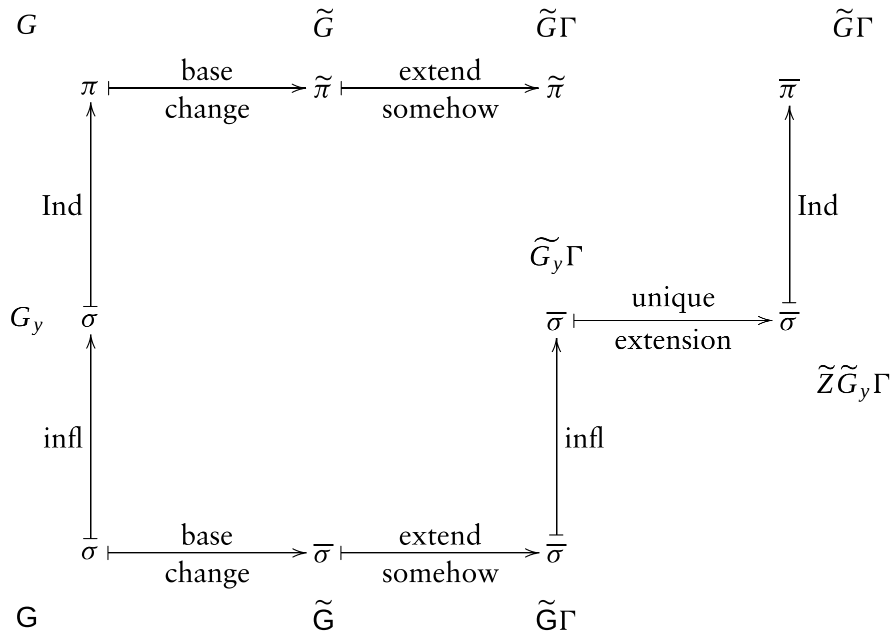
(3) $\psi_1 = \eta \cdot |\cdot|^{\pm 1/2}$, and $\eta|F^\times = \mathrm{sgn}_{E/F}$. Then π has two components $\pi^{(s)}, \pi^{(n)}$, the former being square-integrable, the latter being non-tempered. We will return to this later.

► III. "Stable" supercuspidal

Let $T \subset M_y \simeq U(3)(k_F)$ be a cubic elliptic torus (this means that $\tilde{T} \subset \tilde{M}_y = \mathrm{GL}_3(k_E)$ corresponds to a cubic extension).

Let $\sigma = R_T^{M_y}\theta$ (Deligne-Lusztig representation, θ being in general position). Let $\pi := \mathrm{Ind}_{G_y}^G \inf(\sigma)$.

Restatement of the main theorem in this case:



Then $\tilde{\pi} \simeq \tilde{\pi}'$.

From properties of BC, enough to show

$$(*) \qquad \Theta_{\tilde{\pi}}(g\epsilon) = \Theta_{\tilde{\pi}}(g\epsilon)$$

for all $g \in \tilde{G}$ such that $\mathcal{N}(g)$ is regular.

We can assume that $g \in \tilde{T} \smallsetminus \tilde{Z}\tilde{T}_{0+}(\mathrm{Ker}\,\mathcal{N}|\tilde{T})$, where $\tilde{T}$ is an unramified torus lifting T.

Let $\dot{\theta}_\sigma = \theta_\sigma$ on G_y and 0 on $G \smallsetminus G_y$, $\dot{\theta}_{\tilde{\sigma}} = \theta_{\tilde{\sigma}}$ on $\tilde{Z}\tilde{G}_y\Gamma$, 0 on the rest of $\tilde{G}\Gamma$.

Then the LHS of (*) is

$$\Theta_{\tilde{\pi}}(g\epsilon) = \Theta_{\tilde{\pi},\epsilon}(g) = \Theta_\pi(\mathcal{N}(g)) = \sum_{a \in G_y \backslash G / G_y} c_a \dot{\theta}_\sigma(a^{-1}\mathcal{N}(g)a)$$

(all $a \neq 1$ terms vanish). So this is $\Theta_\sigma(\mathcal{N}(g)) = \Theta_{\tilde{\sigma}}(g\epsilon)$. The RHS of (*) is

$$\Theta_{\tilde{\pi}'}(g\epsilon) = \sum_{a \in \tilde{Z}\tilde{G}_y\Gamma \backslash \tilde{G}\Gamma / \tilde{Z}\tilde{G}_y\Gamma} \tilde{c}_a \dot{\Theta}_{\tilde{\sigma}}(a^{-1}g\epsilon a)$$

[look at action on building, $a \neq 1$ terms vanish], we get $\Theta_{\tilde{\sigma}}(g\epsilon)$. ∎

► **IV. All other cases.**

(A) Supercuspidal representations induced from $R_T^{M_y}\theta$, T not cubic.

(B) Reducible principal series in "case 3" from before.

(C) Supercuspidal representations induced from cuspidal unipotent representations of M_y or representaitons of M_z.

These cases are all linked.

Look at (A) (and most of (C)). Then $T \simeq (k_E^1)^3$ and $\tilde{T}$ is split. Let $\chi = (\chi_1, \chi_2, \chi_3)$ be a character of T. Let χ' and χ'' be cyclic permutations of χ. So we have 4 representations of G, induced from $R_T^{M_y}\chi$, $R_T^{M_z}\chi$, $R_T^{M_z}\chi'$, $R_T^{M_z}\chi''$. This is an L-packet Π. Its base change is the principal series coming from $\chi \circ \mathcal{N}$, suitably extended.

Look at (B) . π had components $\pi^{(s)}$ and $\pi^{(n)}$. The former is in an L-packet with a supercuspidal, and the latter form an L-packet by itself. The base change of these packets are degenerate principal series representation.