Level zero adic character sheaves

A.-M. AUBERT

June 11, 2003

► **(I) A few words on Lusztig's theory of character sheaves.**

This is "at the level of the residue field".

References. Lustizg: character sheaves I–V in Adv. in Math., and Annals. Shoji.
The notes by Mars and Springer on Asterisque.
Aubert-Cunningham: introductory paper in Dubrovnik conference on Functional analysis VII.

► Work over $\bar{\mathbb{F}}_q$. The character sheaves are certain perverse sheaves on $\mathbb{G}$ (algebraic reductive group over $\bar{\mathbb{F}}_q$, connected, Waldspurger has considered the non-connected case for $O(n)$ etc).

Cuspidal character sheaves. If we have $P = LU \subset G$, then we have parabolic restriction of sheaves $\mathrm{Res}_L^G = \nu_! i^*$, where $i : P \hookrightarrow G$, and $\nu : P \twoheadrightarrow L$.

Parabolic induction is more involved, it is adjoint to the restriction:

$$\mathrm{Hom}_{\mathcal{D}L}(\mathrm{Res}_L^G A, A') \simeq \mathrm{Hom}_{\mathcal{D}G}(A, \mathrm{Ind}_L^G A'),$$

where $\mathcal{D} = D_c^b$ is the bounded derived categories of ℓ-adic sheaves, which contains the subcategroy $\mathcal{M}(-)$ of perverse sheaves.

The character sheaves are certain simple objects in $\mathcal{M}(G)$.

► Introduce the category $\mathcal{A}G$, in which the objects are finite sums of character sheaves, and the morphisms are morphisms in $\mathcal{M}(G)$.

Mackey formula. $\mathrm{Res}_L^G \circ \mathrm{Ind}_M^G A = \sum_{g \in L\backslash G/M} \mathrm{ind}_{L\cap {}^gM}^L \circ g \circ \mathrm{Res}_{M\cap L^g}^M A$, where $L^g = g^{-1}Lg$, ${}^gM = gMg^{-1}$.

Duality. $\bigoplus(-1)\,\mathrm{Ind}_L^G \circ \mathrm{Res}_L^G \ldots = \ldots.$

Equivariance: We say that A is G-equivariant if for $m : G \times G \to G$, $\mathrm{pr}_i : G \times G \to G$, we have $\mathrm{pr}_2^* A \simeq m^* A$.

► From now on, assume that G is defined over $\mathbb{F}_q$, hence we have a Frobenius map $F : x \mapsto x^q$.
We say that A is *F-stable* if $\phi_A : F^*A \simeq A$.

$G^F = G(\mathbb{F}_q)$ is a finite group of Lie type. Its space of class functions has a basis $\mathcal{B}_{rep}$ of characters of irreducible representations. There is also a basis B_{geom}, from Frobenius trace χ_{A,ϕ_A} of F-stable character sheaves (almost characters).

Theorem. (Achar-Aubert) Let A be a character sheaf which does not vanish on the unipotent variety of G, then there exists a unipotent class $\mathcal{O}_A$ on G such that $A|\mathcal{O}_A \neq 0$, and if $\mathcal{O}'$ is a unipotent class such that $A|\mathcal{O}' \neq 0$, then $\mathcal{O}' \leq \mathcal{O}_A$ (require p to be good for G).

Corollary. (A-A) Same thing is true if we replace A by an irreducible character of G^F.

It seems to be difficult to prove this directly.

► **(II) Level** 0 **situation.** (with C. Cummingham)

Let G be simply-connected, connected, and semi-simple over $\tilde{k}$, the maximal unramified extension of a local non-archimedean field k with finite residue field $\mathbb{F}_q$.

Denote by $I(G)$ the building of $G(\tilde{k})$. Let i, j etc be facets on $I(G)$, and write $i \leq j$ if i is in the closure of j.

Let $\bar{G}_i$ be the reductive quotient of $G(\tilde{k})_i$.

Definition. Define a category $\mathcal{A}\bar{G}$ as follows: the objects are pairs (A, a), where $A = (A_i)_i$ is a collection of objects indexed by the facets, with $A_i \in \mathcal{A}\bar{G}_i$; and $a = (a_{ij})$ is a collection such that

$$a_{ij} \in \mathrm{Hom}_{\mathcal{A}\bar{G}_i}(\mathrm{Res}^{\bar{G}_i}_{\bar{G}_j} A_i, A_j)$$

for all $i \leq j$. Moreover,

- a_{ij} are isomorphisms and $a_{jj} = id_{\mathcal{A}_{\bar{G}_j}}$, $a_{ik} = a_{jk} \circ \mathrm{Res}^{\bar{G}_j}_{\bar{G}_k} a_{ij}$ for all $i \leq j \leq k$.

A morphism $(A, a) \to (A', a')$ is given by a collection $\phi = (\phi_i)$ such that $\phi_i : A_i \to A'_i$ and the square diagram with arrows $a_{ij}, a'_{ij}, \phi_j, \mathrm{Res}\,\phi_i$ commutes.

► **Parabolic restriction.**
Assume that we have Levi $L \subset G$, $I(L) \hookrightarrow I(G)$.
Let $(A, a) \in \mathcal{A}\bar{G}$, we form the object $(\mathrm{Res}^{\bar{G}_i}_{\bar{L}_i} A_i, \mathrm{Res}^{\bar{G}_j}_{\bar{L}_j} a_{ij})$, which is in $\mathcal{A}\bar{L}$.

► **Parabolic induction.** (built on Mackey formula)
Given $l \in I(L)$ (facet), there is a facet $l^0 \in I(G)$ such that $l \leq l^0$ and $G(\tilde{k})_{l^0} \cap L = L(\tilde{k})_l$.
Let $B \in \mathcal{A}\bar{L}$, $i \in I(G)$, $A_i = \sum_{g \in G_i \backslash G/P, g.i \leq l^{\circ}_{g.i}} \big(g \circ \mathrm{Ind}^{\bar{G}_{g.i}}_{\bar{G}_{l^0_{g.i}}}(B_{l_{g.i}})\big)$.

► Next, we consider Frobenius-stable objects, etc. The main result:

Theorem. Let π be a supercuspidal representation of level 0 of $G(k)$, and let $\bar{\pi}_i := \mathrm{Res}^{G}_{G(k)_i} \pi$. Then there is a unique element $K^\pi \in K\bar{G}_k \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_\ell$ such that

$$\chi_{(K^\pi)_i} = \mathrm{Tr}\,\bar{\pi}_i,$$

for all $i \in I(G, k)$, and K^π is cuspidal.

[The proof involves the Schneider-Stuhler formula].