

An introduction to explicit Bruhat-Tits theory

JIU-KANG YU

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- ▶ This is a slightly expanded note of my talk.
- ▶ I will talk about recent efforts to make Bruhat-Tits theory more explicit and accessible. These are joint works with W. Gan and G. Prasad. I also hope that this talk can serve as a preparation for DeBacker's talk.

By an explicit Bruhat-Tits theory for a p -adic group G , I mean:

- Describe the building $\mathcal{B}(G)$ as a topological space.
- Describe the structure of apartments and (poly)-simplicial complexes on $\mathcal{B}(G)$.
- Describe the parahoric subgroups their Bruhat-Tits schemes, and their Moy-Prasad filtrations.

All these descriptions should be given in terms of some geometry relevant to G .

- ▶ Instead of explaining the results of Gan-Yu (focused on exceptional groups, which are not treated “explicitly” in the canon of Bruhat-Tits) and Prasad-Yu, I will work through an example of classical group. The reasons to do this are

- The treatment of Bruhat-Tits (BSMF 1987) does not cover the simplicial structure in detail.
- The book of Garret is not quite group theoretic. In fact, he started with, say symplectic geometry, and then proceed to construct some simplicial complex, show that it is a building, etc. There is no explicit connection with the viewpoint of Bruhat-Tits theory.
- With the ideas developed in Gan-Yu, now it is possible to figure out everything about the building of classical group fairly easily. The outline given here seems much simpler than the above two references.
- I hope that this lecture is helpful if you will read Gan-Yu, in particular you are not one of the few people, like Wee-Teck, who find exceptional groups more familiar than classical groups.

- ▶ **Setting up.** Let k be a local field of residue characteristic $\neq 2$, (V, \langle, \rangle) a symplectic space over k with symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$. Let $G = \mathrm{Sp}(V)$.

- ▶ **The building of $\mathrm{GL}(V)$.** It is well-known (due to Iwahori-Matsumoto-Bruhat-Tits) that $\mathcal{B} \mathrm{GL}(V)$ is the space of norms on V . This is the starting point of any explicit Bruhat-Tits theory.

We recall the definition of a *norm* α on V : it is a function $\alpha : V \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying

- $\alpha(x + y) \geq \inf\{\alpha(x), \alpha(y)\}$, for all $x, y \in V$;
- $\alpha(\lambda x) = \mathrm{ord}(\lambda) + \alpha(x)$ for all $\lambda \in k, x \in V$;
- $\alpha(x) = +\infty$ if and only if $x = 0$.

(so that $e^{-\alpha}$ is similar to a norm on a real normed vector space).

Example. Let $m = 2n$, v_1, \dots, v_m a basis of V and $c_1, \dots, c_m \in \mathbb{R}$, then

$$\alpha : V \rightarrow \mathbb{R} \cup \{+\infty\},$$

$$\sum \lambda_i v_i \mapsto \inf\{\text{ord}(\lambda_i) + c_i : 1 \leq i \leq m\}$$

is a norm. By varying the c_i 's, we get a space of norms parametrized by \mathbb{R}^m . This Euclidean space is just an apartment on $\mathcal{B} \text{GL}(V)$.

Example. Let L be a lattice in V , define $\alpha = \alpha_L$ by

$$\alpha(x) = \sup\{t : x \in \pi^t L\}.$$

Then α is a norm. Such norms are precisely the vertices on $\mathcal{B} \text{GL}(V)$.

Fact. Since $\text{GL}(V) \simeq \text{GL}(V^*)$, we have $\mathcal{B} \text{GL}(V) \simeq \mathcal{B} \text{GL}(V^*)$. This just means that every norm α on V determines a *dual norm* on the dual space V^* .

► **The topological space of $\mathcal{B} \text{Sp}(V)$.** There is an involution σ of $\text{GL}(V)$ such that $G = \text{GL}(V)^\sigma$. The involution σ acts on $\mathcal{B} \text{GL}(V)$ by transport of structure, and $\sigma(\alpha)$ just identifies the dual norm on V^* back to a norm on V via $V^* \simeq V$ offered by the duality \langle, \rangle .

By a theorem of Prasad-Yu, $\mathcal{B} \text{Sp}(V) = (\mathcal{B} \text{GL}(V))^\sigma$. So,

THEOREM 1. $\mathcal{B} \text{Sp}(V)$ is the space of self-dual norm on V .

This description was first conjectured by Weil in 1963, announced as a theorem by Bruhat-Tits in 1974, proved by them in 1987. Recently, Moy and Kim re-discovered the statement and gave a nice proof.

As mentioned above, this now follows from a general theorem of Prasad-Yu. However, it is also very easy to give a one-page proof using a general formalism of Gan-Yu. One only needs a single arithmetic input: the fact that any two self-dual lattice in V are conjugate by $G(k)$. [*Remark.* The proof of Gan-Yu or Bruhat-Tits is valid for $p = 2$ also].

► Theorem 1 is very useful. But it doesn't answer questions as: can one give some explicit points on $\mathcal{B}G$? how do we know whether a point is a vertex? etc.

The rest of this talk is to answer these questions with minimal amount of computations. I would like to emphasize that Theorem 1 is the key point. It is why we can give a much simpler treatment of the whole theory.

► **The root system.** The minimal computation we need to do is to work with a torus and compute the affine root system, so that we know what an apartment is like.

We first specify a maximal torus: $S = \{s(t_1, \dots, t_n) : t_i \in \mathbb{G}_m\}$, where $s(\vec{t}) \cdot e_i = t_i e_i$, $s(\vec{t}) \cdot f_i = t_i^{-1} f_i$, for $1 \leq i \leq n$.

Let a_i be the character of S such that $a_i(s(\vec{t})) = t_i$. Then

$$\Phi = \Phi(G, S) = \{\pm 2a_i, \pm a_i \pm a_j\}$$

is of type C_n .

The root subgroups are easy to describe: for example, for $a = a_i - a_j$, $U_a = \{x_a(u) : u \in \mathbb{G}_a\}$, where $x_a(u)$ acts on $\langle e_i, e_j \rangle$ by $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and on $\langle f_i, f_j \rangle$ by $\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}$ (and fixing the other e_k 's, f_k 's).

For $a = 2a_i$, U_a acts on $\langle e_i, f_i \rangle$ by $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, etc.

► **The affine root system.** Of course, what we have written down is just a Chevalley system $\{x_a\}_{a \in \Phi}$ for the split group (G, S) . It is a general fact that whenever you have a Chevalley system (more generally a Chevalley-Steinberg system), you know:

- there is a (hyperspecial) point y_0 on the building (actually on the apartment $A(G, S)$), determined by the Chevalley system.
- using y_0 as the origin of $A(G, S)$, the affine root system is

$$\{\mathbb{Z} \pm 2a_i, \mathbb{Z} \pm a_i \pm a_j\}.$$

- the filtration groups of the root subgroups are as follows: $U_a(k)_{y_0, r} = \{x_a(u) : u \in k, \text{ord}(u) \geq r\}$ for any a .

► **Finding a set of standard vertices.** Let us first identify y_0 in the description of Theorem 1. By definition, the hyperspecial subgroup $G_{y_0, 0}$ associated to y_0 is generated by $S(\mathcal{O}_k)$ and $U_a(k)_{y_0, 0} = x_a(\mathcal{O}_k)$, $a \in \Phi$. Clearly, the matrices of all these generating elements have integral coefficients (relative to the basis $e_1, \dots, e_n, f_1, \dots, f_n$). So it is clear that $G_{y_0, 0}$ fixes the lattice L_0 spanned by

$$e_1, \dots, e_n, f_1, \dots, f_n.$$

The action of σ on $\mathcal{B} \text{GL}(V)$ is such that $\sigma(\alpha_L) = \alpha_{L^\perp}$, where

$$L^\perp = \{x \in V : \langle x, L \rangle \subset \mathcal{O}_k\}$$

is the lattice dual to L . Since L_0 is self-dual, α_{L_0} lies on $\mathcal{B} \text{Sp}(V)$, and it must be the unique point y_0 fixed by $G_{y_0, 0}$.

Other vertices. Now a system of simple affine roots can be choose to be

$$\{\alpha_0, \dots, \alpha_n\} = \{1 - 2a_1, a_1 - a_2, \dots, a_{n-1} - a_n, 2a_n\}.$$

We notice that $\alpha_0 + 2(\alpha_1 + \dots + \alpha_{n-2}) + \alpha_n = 1$. Setting $(\alpha_0, \dots, \alpha_n)$ to $(1, 0, \dots, 0)$, we get the vertex y_0 . Setting it to $(0, 1/2, 0, \dots, 0)$, we get another vertex y_1 . The vector $y_1 - y_0$ lies in $\text{Hom}(\mathbb{G}_m, S) \otimes \mathbb{R} \subset \text{Hom}(\mathbb{G}_m, T) \otimes \mathbb{R}$, which acts on $A(\text{GL}(V), T)$, where T is the maximal k -split torus of $\text{GL}(V)$ determined by the basis e_i, f_i . We know how this action works in $\mathcal{B} \text{GL}(V)$, therefore, we can figure out what y_1 is.

In more details, y_1 is obtained from y_0 by applying the translation $(1/2, 0, \dots, 0, -1/2, 0, \dots, 0)$ to $A(\text{GL}(V), T) \simeq \mathbb{R}^{2n}$. If we are translating by $(1, 0, \dots, 0, -1, 0, \dots, 0)$, it is the same as the action of $(\pi, 1, \dots, 1, \pi^{-1}, 1, \dots, 1) \in T(k)$ on y_0 , hence is the lattice spanned by $\{\pi e_1, e_2, \dots, e_n, \pi^{-1} f_1, f_2, \dots, f_n\}$.

Now it is easy to see that y_1 is the mid-point of L_1 and L_1^\perp , where L_1 is the lattice spanned by

$$\pi e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n.$$

Hence L_1^\perp is the lattice spanned by

$$e_1, e_2, \dots, e_n, \pi^{-1} f_1, f_2, \dots, f_n.$$

Similarly, we can find all vertices in the chamber determined by the above system of simple affine roots. They are the mid-points y_i of L_i and L_i^\perp , where L_i is the lattice spanned by

$$\pi e_1, \dots, \pi e_i, e_{i+1}, \dots, e_n, f_1, \dots, f_n.$$

The point y_n is hyperspecial.

► **Identifying all vertices.** What is special about the lattice L_i ? One can try to find intrinsic properties of them (invariant by $G(k)$). Then it is natural to conjecture the following description of vertices on $\mathcal{B}\mathrm{Sp}(V)$: *there is a bijection*

$$\{\text{vertices on } \mathcal{B}\mathrm{SO}(V)\} \leftrightarrow \{\text{lattices } L \text{ in } V \text{ satisfying } \pi L^\perp \subset L \subset L^\perp\}.$$

A vertex “ L ” is of “type i ” (in the sense that it is conjugate to y_i) if and only if $\dim L^\perp/L = 2i$.

THEOREM 2. *This description is correct.*

This theorem amounts to the following: a lattice L satisfying $\pi L^\perp \subset L \subset L^\perp$ is $G(k)$ -conjugate to our standard examples, or equivalently, such an L admits a suitable Witt-type decomposition.

This Witt-type decomposition theorem is proved in Bruhat-Tits or Garret (and some work of Waldspurger can be considered as a reference too). Though it is not very hard, it is not easy and the proof is long. Here is a very simple argument, based on ideas developed in Gan-Yu.

We first explicate the correspondence $L \mapsto$ a point on $\mathcal{B}G$. Of course, it is just $L \mapsto v_L =$ the mid-point of L and L^\perp . When we write down this point as a norm, we realize that it takes values in $\frac{1}{2}\mathbb{Z}$ (in fact, this norm α takes value 0 on $L \setminus \pi L^\perp$, value $\frac{1}{2}$ on $\pi L^\perp \setminus \pi L$, etc.)

Now we assert that a self-dual norm α takes values in $\frac{1}{2}\mathbb{Z}$ if and only if α is a vertex on $\mathcal{B}G$. It is clear that this statement can be checked by computing in an apartment, and there we check it easily.

The theorem is clear from these two observations. Notice that this simple argument recovers the Witt-type decomposition theorem. This argument relies on Theorem 1 in a crucial way. In the paper of Bruhat-Tits, the Witt-type decomposition is needed to prove Theorem 1. However, we now have several easier proofs of Theorem 1 independently.

► **The maximal parahoric subgroups.** Without giving details of justification, I will mention the following results, which are at least very plausible.

Let L be a lattice determining a vertex y . Then the parahoric subgroup $G_{y,0}$ acts on L^\perp/L and $L/\pi L^\perp$. We notice that these are vector spaces over the residue field κ of k , and they carry canonical non-degenerate symplectic forms which are preserved by the action of $G_{y,0}$. Therefore, we get a map $G_{y,0} \rightarrow \mathrm{Sp}(L^\perp/L) \times \mathrm{Sp}(L/\pi L^\perp)$.

In fact, the image is $\mathrm{Sp}(L^\perp/L) \times \mathrm{Sp}(L/\pi L^\perp)$, and this is the maximal reductive quotient of $\underline{G}_y \otimes \kappa$. Moreover,

THEOREM 3. (BRUHAT-TITS) \underline{G}_y is the schematic closure of G in $\mathrm{GL}(L) \times \mathrm{GL}(L^\perp)$.

► **Incidence relations.**

THEOREM 4. *Let L, L' be two lattices which correspond to vertices on $\mathcal{B}G$. Then L is incident to L' if and only if $L \subset L'$ or $L' \subset L$.*

To see this, we can assume that the two vertices lie on a standard apartment. Then the statement is a simple exercise in linear programming (fixing L , the points $v_{L'}$ with $L' \subset L$ lies in a bounded subset of the apartment defined by linear inequalities, and we can check that the only vertices in the bounded sets are those adjacent to v_L).

Consequently, to specify a chamber is to give a chain of lattices $M_0 \subsetneq \dots \subsetneq M_n$ such that $\pi M_i^\perp \subset M_i \subset M_i^\perp$ for each i . Since all chambers are $G(k)$ -conjugate to each other, we can deduce that any such chain of lattices is $G(k)$ -conjugate to the standard one, in other words, they admit a “simultaneous Witt-type decomposition.” Again, this type of result is needed before proving Theorem 1 in earlier approaches. But we get it for free.

► **Other classical groups**

One can treat all classical group (defined by an involution) in the same way. The description of the points on the topological space is uniform: set of self-dual norms. But just as we can only describe the root systems uniformly for classical groups in a fixed series, we need to treat the simplicial structures of classical groups in different Witt towers separately.

As an example, I will describe the vertices in a split even orthogonal group $\text{SO}(V)$, $\dim V = 2n$, $n \geq 4$. They are in bijection with the set of lattices L satisfying

$$(i) \pi L^\perp \subset L \subset L^\perp, \quad (ii) \text{length}(L^\perp/L) \neq 2, 2n - 2.$$

Remark. The set of L satisfying (i) is precisely the set of L whose stabilizer is a maximal compact subgroup of $\text{SO}(V)$.

Two vertices L and L' are incident if and only if

- $L \subset L'$, or
- $L' \subset L$, or
- $\text{length}(L^\perp) = \text{length}(L'^\perp/L') = 0$, $\text{length}(L/(L' \cap L)) = \text{length}(L'/(L \cap L')) = 1$, or
- $\text{length}(L^\perp) = \text{length}(L'^\perp/L') = 2n$, $\text{length}((L' + L)/L) = \text{length}((L' + L)/L') = 1$.

Remark. I emphasize that all you have to do is start with the analogue of Theorem 1 for the orthogonal group, and go through the line of arguments for $\text{Sp}(V)$ above. It only involves linear geometry on a real affine space. We don't need to solve any arithmetic/algebra problems about lattices in quadratic spaces.

In contrast, the problem of describing the simplicial complex of a quasi-simple algebraic group (say a classical group) is sometimes more involved. Although the analogue of Theorem 1 is still true, the remaining work involves spherical geometry rather than linear geometry.