UNREFINED MINIMAL K-TYPES FOR CLASSICAL GROUPS

(AN EXERCISE IN MOY-PRASAD THEORY)

Jiu-Kang Yu

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- F: non-archimedean, local compact field.

For $G = \mathrm{GL}_n(F)$,

$$\text{unrefined minimal } K\text{-types} = \text{fundamental strata}.$$

This is a collection $\mathcal{F}$ of certain pairs (K, ρ), where $K \subset G$ is compact open, $\rho \in G^\wedge$.

By a theorem of Bushnell and Howe-Moy, for every irr. smooth rep'n π, $\exists$ at least one $(K, \rho) \in \mathcal{F}$ s.t. $\pi|K \supset \rho$.

$\rightsquigarrow$ a crude classification of irr. smooth rep'ns of G.

The theory of Bushnell-Kutzko refines this classification by enriching a fundamental stratum into a "semisimple type".

- Moy-Prasad theory = generalization of the first crude step to other connected reductive groups.

This talk: an exercise of working out MP theory for classical groups (groups of symmetry for hermitian forms).

- There are already the theories of Morris, Stevens, Kariyama-Miyauchi providing similar results of "fundamental stratas". So why this exercise?

▶ **Cons.**

- Laborious. There are 18 infinite families, hence 18 cases.

- The work of Morris-Stevens-Kariyama-Miyauchi is in a language close to that of Bushnell-Kutzko. It may be easier to implement Bushnell-Kutzko's theory from this approach. Indeed, more progress has been made.

- Each case has to be deciphered individually. The result is not quite uniform. In contrast, the M-S-K-M results are more or less uniform in cases that they apply.

▶ **Pros.**

- It is routine.
- Greater generality: all classical groups, groups isogenous to classical groups.
- Somewhat more refined results in each case.
- Usability in other questions: e.g. the work of Adler, Debacker, Moy, Prasad.

▶ Elements of Moy-Prasad theory: • *$\mathcal{B}G$ as a simplical complex*, • $G_{x,r}$, • a (finite) set Q of optimal points.

Remark. We need to know a simplex on $\mathcal{B}G$ (the Bruhat-Tits building of G) to talk about a set of optimal points.

The main theorem of Moy-Prasad theory says the following: *The collection of pairs:*

$$\{(G_{x,r}, \rho) : x \in Q, G_{x,r} \neq G_{x,r+}, \rho : \textit{non-degenerate character}\}$$

play the role of "fundamental strata" a.k.a "unrefined minimal K-types".

Some definitions will be recalled later.

Remark. In this talk, we concentrate on the optimal points. This is because the first two elements are known:

Let G be a classical group G with standard representation $G \hookrightarrow H = \mathrm{GL}_D(V)$, so that $G = (H^{\phi})^{\circ}$ for some involution ϕ. Assume that the residue characteristic of F is not 2.

- There is a simple description of $\mathcal{B}G$ *as a topological space*: $\mathcal{B}G = (\mathcal{B}H)^{\phi}$ (Bruhat-Tits, Kim-Moy, Prasad-Yu or Gan-Yu).

There is a recipe in the work of Gan-Yu that calculates the simplicial structure easily from the above result (cf. my lecture at Banff 2001, notes available at `http://www.math.umd.edu/˜ yu/notes`.

- We have $G_{x,r} = H_{\iota x,r} \cap G$ for a suitable map $\iota : \mathcal{B}G \to \mathcal{B}H$. On the other hand, $H_{y,r} = \mathrm{GL}_D(V)_{y,r}$ is easy to work out: they are groups associated to *lattice sequences* in the language of Bushnell-Kutzko (see e.g. the work of Broussous).

▶ We now begin to work out the set of optimal points. For simplicity, assume that G is split.

After choosing a (standard) maximal F-split torus S, a chamber C on $A(S)$, we have the data Φ, Φ^{aff}. Let

$$\Sigma = \{\phi \in \Phi^{\mathrm{aff}} : \phi > 0 \text{ and } 1 - \phi > 0\}.$$

Let $\mathcal{S}$ be a subset of Σ. We define an $\mathbb{R} \cup \{-\infty\}$-valued function on $\bar{C}$:

$$f_{\mathcal{S}}(y) = \inf\{\phi(y) : \phi \in \mathcal{S}\}.$$

Definition. (Moy-Prasad) (i) An *optimal point* for $\mathcal{S}$ is a point $y \in \bar{C}$ where $f_{\mathcal{S}}$ achieves its maximal value. (ii) A finite subset Q of $\bar{C}$ is called *a set of optimal points* if for every $\mathcal{S} \subset \Sigma$, we can find $y \in Q$ such that y is an optimal point for $\mathcal{S}$.

Remark. Call $\mathcal{S}$ *reduced* if no two elements of $\mathcal{S}$ is comparable. It is easy to see that definition (ii) is unchanged if we replace for every $\mathcal{S} \subset \Sigma$ by for every reduced $\mathcal{S} \subset \Sigma$. Hence we will concentrate on the reduced sets.

► A model example: $G = \mathrm{GL}_{n+1}(\mathrm{F})$, of type A_n. Let $N = n + 1$, $a_0, \ldots, a_{N-1}$ be the simple affine roots. Then it is easy to verify that Σ consists of precisely of

$$\phi_{i,s} = \underbrace{a_i + a_{i+1} + \cdots + a_{i+s-1}}_{s \text{ terms}},$$

where $i = 0, \ldots, N-1$, $s = 1, \ldots, N-1$, and the subscripts of a are considered as in $\mathbb{Z}/N\mathbb{Z}$.

We picture $\phi_{i,s}$ as a string of consecutive elements in $\mathbb{Z}/N\mathbb{Z}$. Then it is clear

Lemma. If $\mathcal{S} \subset \Sigma$ is reduced, then $\#\mathcal{S} \leq N$, with equality precisely when $\mathcal{S} = \{\phi_{0,s}, \phi_{1,s}, \ldots, \phi_{N-1,s}\}$ for some s.

When $\#\mathcal{S} = N$, it is easy to see that the barycenter of C is the unique optimal point of $\mathcal{S}$. On the other hand, notice

Lemma. If $\#\mathcal{S} < n + 1$, then there is an optimal point for $\mathcal{S}$ on ∂C, the boundary of C.

PROOF. Let V be the real vector space underlying the affine space A. Since $\#S \leq n = \dim V$, we can find a non-zero $v \in V$ such that $d\phi(v) \geq 0$ for all $\phi \in \mathcal{S}$.

Let $y \in \bar{C}$ be any optimal point for $\mathcal{S}$. Then for any $t \geq 0$, $y + tv$ is also optimal for $\mathcal{S}$ as long as it lies in $\bar{C}$. ∎

Say, $\mathcal{S}$ has an optimal point y on the facet F defined by $a_{N-1} = 0$. Then y is also an optimal point for the system $\mathcal{S}|_F$ in an obvious sense.

Notice that $\Sigma|_F$ can be described exactly like Σ, with n replaced by $n - 1$. Thus by induction we have shown

Proposition. The set Q of barycenters of facets of C is a set of optimal points.

Observation. This is a *canonical* set of optimal points in the sense that it is the *smallest*: $Q \subset Q'$ for any other set Q' of optimal points.

Lattice theoretical interpretation.

Let y is the barycenter of a facet F, and L_1, …, L_e be lattices corresponding to the vertices of F. Then $\{\pi^i L_j : i \in \mathbb{Z}, j = 1, \ldots, e\}$ is a lattice chain of period e, and $G_{y,k/e} = 1 + \mathfrak{P}^k$, where $\mathfrak{P}$ is the radical of

$$\bigcap_j \mathrm{End}(L_j).$$

Thus we recover the theorem of Bushnell and Howe-Moy.

► The case of $G = \mathrm{Sp}_{2n}(F)$.

Weighted barycenter. Let $P_0, \dots, P_s$ be points in an affine space, and w_0, …, w_s be weights (non-negative real numbers, not all zero). Then we can form a weighted barycenter:

$$Q = \frac{1}{w} \sum_{i=0}^{s} w_i P_i,$$

where $w = \sum_{i=0}^{s} w_i$.

Now let $G = \mathrm{Sp}_{2n}$ and form Φ^{aff}, Σ, etc. accordingly. For each facet F of C, with vertices $P_0, \dots, P_s$, we form a weighted barycenter Q_F by giving P_i weight 1 if it is hyperspecial, weight 2 otherwise.

Theorem. The set

$$\{Q_F : F \text{ is a facet of } C\}$$

is a set of optimal points. Moreover, it is canonical in the sense that it is the smallest.

Lattice theoretical interpretation.

Let $e_1, \dots, e_n$, $f_1, \dots, f_n$ be a symplectic basis of the underlying symplectic vector space. For $i = 0, \dots, n$, let L_i be the $\mathcal{O}_F$-span of

$$\pi e_1, \dots, \pi e_i, e_{i+1}, \dots, e_n,\ f_1, \dots, f_n.$$

Then in the embedding of $\mathcal{B}\,\mathrm{Sp}_{2n}(F) \hookrightarrow \mathcal{B}\,\mathrm{GL}_{2n}(F)$,

$$\text{the two hyperspecial vertices of } C \mapsto L_0 \text{ and } L_n,$$
$$\text{the others vertices} \mapsto \text{the midpoint of } L_i \text{ and } L_i^{\perp}, \qquad i = 1, \dots, n-1$$

(see e.g.. my Banff lecture notes).

Thus $y = Q_F \mapsto$ the *ordinary* barycenter of a collection of distinct lattices among $\{L_0, \dots, L_n, L_1^{\perp}, \dots, L_{n-1}^{\perp}\}$. We can form a lattice chain from these. The only feature of lattice chains obtained this way is that it is *self-dual* (if L is in the chain, so is $L^{\perp}$).

The Moy-Prasad group $G_{y,r}$ is simply $\mathrm{Sp}_{2n}(F) \cap (1 + \mathfrak{P}^k)$ for a suitable k. Therefore, *to form the theory of fundamental strata for symplectic groups, all we need are these* $\mathrm{Sp}_{2n}(F) \cap (1 + \mathfrak{P}^k)$ *associated to self-dual lattice chains.*

► Sketch of the calculation. Say $n = 3$, and a, b, c, d are the simple affine roots, so that $a + 2b + 2c + d = 1$.

Then Σ consists of

$$\begin{gathered} a, b, c, d, \\ a + b, b + c, c + d, \\ a + 2b, a + b + c, b + c + d, 2c + d, \\ a + 2b + c, a + b + c + d, b + 2c + d, \\ a + 2b + 2c, a + 2b + c + d, a + b + 2c + d, 2b + 2c + d. \end{gathered}$$

We can picture these as substrings of the string $dcbabcdcba$.

Thus the aboves are

$$\begin{gathered} a, b, c, d, \\ ab, bc, cd, \\ bab, abc, bcd, cdc, \\ babc, abcd, bcdc, \\ cbabc, babcd, abcdc, bcdcb. \end{gathered}$$

We will be able to reduce any $\mathcal{S}$ with $\#\mathcal{S} < n + 1 = 4$ to a case of lower rank by induction. The reduced sets of length ≥ 4 are precisely the rows 1, 3, 5.

The unique optimal point for these reduced sets is

$$\begin{aligned} (a, b, c, d) =& \frac{1}{6}(1, 1, 1, 1) \\ &= \text{weighted barycenter of} \\ &\quad (1, 0, 0, 0), (0, \frac{1}{2}, 0, 0), (0, 0, \frac{1}{2}, 0), (0, 0, 0, 1) \\ &\quad \text{with weights } 1, 2, 2, 1. \end{aligned}$$

Remember that we are working with the example of type C_3. As mentioned above, when an optimal point happens on a facet of ∂C, we are reduced to a case of lower rank.

For example, if this happens on the wall $b = 0$, by looking at $\Sigma|_{b=0}$, we see that we end up in a case of type C_2.

But when the optimal poin is on the wall $a = 0$, the problem of studying $\Sigma|_{a=0}$ is not the one occurred in type C_2. But it is the one occurred in type C-B_2 (notations are that of Tits' Corvallis table).

In general, the study of type C_n requires solving type C-B_n and type C-BC_n^{II} first. The analysis for them are similar, though more involved. In my solution, a facet may contain more than one optimal points (up to 4).

► The cases of SO_{2n+1} and SO_{2n} are even more complicated. In my solution, a facet may contain more than one optimal points (up to 9). But in all cases, the lattice theoretical interpretation is quite neat. In some cases, we do need lattice sequences, not just lattice chains.

Question. Is there always a *smallest* solution? It is so in many cases but I didn't try to check all cases. On the other hand, I do not know a case that this fails. Nor do I know a way to reason the existence of a smallest solution in general. If this is true in all cases, this smallest set must have some significance.