On some interesting quasi-split subgroups

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► Let k be a non-archimedean local field, K its maximal unramified extension, p the characteristic of the residue field.

Let G be an absolutely simple algebraic group over k.

► Let H be a connected reductive subgroup in G.

Question. When can we have an embedding $\mathcal{B}(H/k) \hookrightarrow \mathcal{B}(G/k)$?

This was answered by E. Landvogt. In fact, one can embed $\mathcal{B}(G/K) \hookrightarrow \mathcal{B}(G/K)$ in a Galois equivariant way.

Question. Uniqueness of embedding? Do we have $H_{x,r} = G_{x,r} \cap H$ for some embedding(s)?

Landvogt claimed some uniqueness results that are not true. For example, it is easy to see that for $\mathrm{SL}_2 \hookrightarrow \mathrm{SL}_3$ with $p = 2$, the embedding is not unique. More generally, we have non-uniqueness examples of $\mathrm{SL}_n \hookrightarrow \mathrm{SL}_{n^2-1}$ (adjoint representation), $p|n$.

Remark. The uniqueness does hold when $p = 0$. For example, this applies to the geometric Langlands case. A proof different from Landvogt is needed, due to Prasad but can be obtained from Yu.

► Let H be a maximal elliptic torus. The answer to the above question is still unknown. But it can be answered affirmatively when H is tamely ramified: over L/k a tamely ramified splitting field, we have $\mathcal{B}(H/L) \hookrightarrow \mathcal{B}(G/L)$ as an apartment, and we get $\{x\} = A(H, L)^\Gamma \hookrightarrow \mathcal{B}(G/L)^\Gamma = \mathcal{B}(G/k)$, where $\Gamma = \mathrm{Gal}(L/k)$ and we have used Rousseau's theorem.

A conjecture. In general (L/k not tame), we have $\{y\} = A(H, L)^\Gamma \hookrightarrow \mathcal{B}(G/L)^\Gamma \supset \mathcal{B}(G/k)$, then maybe we can take x to be the point closest to y on $\mathcal{B}(G/k)$.

► Now we will discuss two constructions of quasi-split groups.

Let S be a maximal k-split torus. Let $T \supset S$ be a maximal K-split torus of G defined over k.

Since G/K is quasi-split by Steinberg, $Z(T) := Z$ is a maximal K-torus of G. Let $\Phi = \Phi(G, T)$, and Ψ be the absolute affine root system of G with respect to T. Let I be an Iwahori subgroup of $G(K)$ defined over k such that $I \cap T(K)$ is the maximal bounded subgroup of $T(K)$.

Let $\Delta \subset \Psi$ be the basis determined by I. Then $\Gamma = \mathrm{Gal}(K/k)$ acts on I and Δ.

► Fix a special vertex $\omega \in \Delta$ and let Ω be the Γ-orbit $\Gamma.\omega$ of ω. We want to choose ω so that if Ω is removed from the absolute local Dynkin diagram, the resulting diagram is still connected. This is possible except for inner forms of type A_n.

Now $\Delta \setminus \Omega$ determines a Levi subgroup whose derived group will be denoted by G^*. Then G^* is defined over k since $\Delta \setminus \Omega$ is Galois stable. It can be described as the group generated by $U_{\pm\dot{a}}$ for $a \in \Delta \setminus \Omega$.

Claim. G^* is quasi-split and it splits over K if G does.

Also, if G is simply connected, so is G^*.

PROOF. Let $Z^* = Z \cap G^*$. This together with $U_{+\dot{a}}$, $a \in \Delta \setminus \Omega$, generates a Borel subgroup, which is defined over k. ∎

Since we only removed a "small" number of vertices, this subgroup is fairly large. For example, if G is a (non-quasi-split) even special unitary group (for unramified quadratic extension), G^* is (quasi-split) of the same kind, in 2 less variables.

► **An application.** (with Raghunathan in Annals volume 109), computation of $H_c^2(G(k), \mathbb{R}/\mathbb{Z})$. We showed that the restriction

$$H_c^2(G(k), \mathbb{R}/\mathbb{Z}) \to H_c^2(G^*(K), \mathbb{R}/\mathbb{Z}),$$

is injective except if G is of type C_n. The latter is bounded by $\hat{\mu}(k)$. [The problem of type C_n is that $\Delta \setminus \Omega$ has no long root at all].

► **Deligne's central extensions.** In Lie group theory, there is a result of Bott-Samelson that $\pi_3(\mathcal{G}) \simeq \mathbb{Z}$ for $\mathcal{G}$ a compact simple simply connected group. This implies that π_4 of the classifying space is $\mathbb{Z}$ and the lower π_i's are trivial. Thus $H^4(B\mathcal{G}) = 0$ and the lower H^i's are trivial.

For any field k, Deligne constructed (using the analogue of the above in étale cohomology) for a simply connected, absolutely simple group G/k, an extension

$$1 \to \mu(k) \to E \to G(k) \to 1.$$

We can pull-back to get an extension E' of H for any $H \subset G(k)$. Thus one can ask what is the relation of this with the central extension attached to H itself.

[Go to a field extension over which G and H split. Take a cocharacter λ of G. Let r be the squared length of a cocharacter of a long root of H.]

The functoriality theorem is that one extension is r times the other in the Ext group.

Claim. Deligne's extension is of the maximal possible order in the Ext-group.

PROOF. Let $G^* \subset G$ be the quasi-split group we constructed. Take a long root of G^* and the associated SL_2...

For type C_n, A_n, some modification...

Remark. $\mathcal{B}(G^*/k) \hookrightarrow \mathcal{B}(G/k)$, but not simplicially. But the compatiblity with Moy-Prasad group is OK.

► If $k = \mathbb{R}$, Deligne's central extension is of degree 2. Assume that $G(\mathbb{R})$ is non-compact. Then $\pi_1(G(\mathbb{R}))$ is of order 1 or 2, or is $\mathbb{Z}$. Suppose that π_1 is non-trivial, is Deligne's extension the unique non-trivial 2-sheeted covering of $G(\mathbb{R})$?

It happens that π_1 is nontrivial if and only if the long root spaces are 1-dimensional. Moreover, in this case, the associated map $SL_2(\mathbb{R}) \to G(\mathbb{R})$ induces a surjection on π_1. (These are now proved by Adams and Trapa.)

One can use this to show that if $\pi_1(G(\mathbb{R}))$ is nontrivial, then Deligne's extension of $G(\mathbb{R})$ is indeed its two-sheeted covering (see my paper in Advances in Math).

► **A second construction of quasi-split group.**

Motivation. Consider $SL_n(L) \subset SL_n(D)$, where $L \subset D$ is a totally ramified splitting field.

The details have appeared on Amer. J. Math. volume 105.

This construction gives embedding of buildings which are simplicial.