Explicit matching for level zero discrete series of p-adic simple algebras

W. ZINK

June 10, 2003

► 1. The Jacquet-Langlands map

We will give a description of the Jacquet-Langlands correspondence in terms of types if the representations are of level 0. Jointly with A.J. Silberger.

► Let F be a local field. Fix $n \geq 1$, and consider $A = M_m(D_d)$, where d is the order of A in the Brauer group (degree of division algebra), and where $md = n$. The product $n = md$ is fixed.

Let $R^2(A^\times)$ be the irreducible L^2-representations of $A^\times$ modulo equivalence. Then JL is a bijection between $R^2(D_n^\times)$ and $R^2(A^\times)$, $\Pi^D \mapsto \Pi^A$, where $n = md$. This is characterized by character relation on the regular elements as usual.

A representation V is of level 0 if $V^{1+\mathfrak{p}} \neq 0$, where $\mathfrak{p} = M_m(\mathfrak{p}_D)$.

► 2. Langlands parameters for $R_0^2(A^\times)$.

Here, $R^2(A^\times)$ is the set of discrete series representations of depth zero.

Our Langlands parameter is such that

(a) Start with a Galois parameter
(b) We have $P \mapsto \Pi_P^D$ explicit in D.
(c) Define $\Pi_P^A = JL(\Pi_P^D)$.

(b) is very easy, because at level 0 of $D^\times$, we simply have a representation of $D^\times/U^1(D) \simeq k_D^\times \rtimes \mathbb{Z}$, the first factor being cyclic.

Parameters. Let F_f/F be an unramified extension of degree f such that $f|n$. Let $\chi_f \in X_t(F_f^\times)$, where X_t is the group of *tame* characters (i.e. trivial on $U^1(F_f)$). Assume also that χ_f is *regular* in the sense that the $\mathrm{Gal}(F_f/F)$-orbit of χ_f is of size f.

Let $\mathfrak{J}_0^n$ be the set of Galois orbits of all such χ_f's, for all $f|n$. These data will serve as our parameters.

Proposition. There is a bijection $\mathfrak{J}_0^n$ and $R_0^2(A^\times)$, $[\chi_f] \mapsto \Pi_{\chi_f}^A$, where $\Pi = \Pi_{\chi_f}^A$ is uniquely determined by the formula

$$\Theta_\Pi(x) = (-1)^{m-1} \sum_{\eta \in [\chi_f]} \eta \circ N_{F(x)/F_f}(x)$$

if x is regular elliptic, and $F_f \hookrightarrow F(x)$.

► 3. Level 0 types.

In A, we have the maximal order $\mathfrak{A}_1$, and Jacobson radical $\mathfrak{P}_1$.

Bernstein. $(1 + \mathfrak{P}_1, \mathbf{1})$ is a type for $A^\times$. It corresponds to a finite number of Bernstein components from the Bernstein decomposition.

Thus $\mathcal{M}(A^\times) = \mathcal{M}_0(A^\times) \times (\ldots)$, where the first factor are representations generated by $(1+\mathfrak{P}_1)$-fixed vectors.

(Some overlap with work of L. Morris)

Let $\mathfrak{A} \subset \mathfrak{A}_1$ be a standard hereditary order, i.e. $\mathfrak{A}^\times/(1+\mathfrak{P}_1) \hookrightarrow \mathrm{GL}_m(k_D)$ maps to a standard parabolic. Let τ be an irreducible cuspidal representation of $\overline{\mathfrak{A}^\times} \simeq \mathrm{GL}_{s_1}(k_D) \times \cdots \times \mathrm{GL}_{s_r}(k_D)$, where r is the period, and then $\tau = \sigma_1 \otimes \cdots \otimes \sigma_r$.

Then $(\mathfrak{A}^\times, \tau)$ is a complete set of level 0 type for $A^\times$. [Here $\tau = \bigotimes \sigma_i \sim \tau' = \bigotimes \sigma_i'$ if and only if the collection $\{\sigma_i\}$ and $\{\sigma_i'\}$ are the same (counting multiplicity) modulo the action of $\mathrm{Gal}(k_D/k)$].

Now $(\mathfrak{A}^\times, \tau)$ is contained in a discrete series (Π, V) if and only if $\mathfrak{A} = \mathfrak{A}_r$, $\overline{\mathfrak{A}_r^\times} = (\mathrm{GL}_s(k_D))^r$, and all the factors are Galois conjugate. Following Bushnell-Kutzko, we call these *simple level-*0 *types*. They occur in the discrete series with multiplicity 1.

► **4. Parameters for simple level-0 types.**

Recall that k is the residue field of F, and k_n/k the degree n extension. Let ϕ be a generator of $\mathrm{Gal}(k_n/k)$.

Consider $\langle\phi\rangle\backslash X(k_n^\times)$, and an element $[\bar\chi]$ in this set. Let $f = \#[\bar\chi]$. Fix $n = md$. Then $[\bar\chi]$ determines the following data:

(a) A cuspidal representation $\sigma = \sigma_{\bar\chi,d}$ on $\mathrm{GL}_{f'}(k_D)$, where $f' = f/(f,d)$. We have $\#[\sigma] = (d,\ f)$.

We also get from $\bar\chi$ an equivalence class $[\tau]_{\bar\chi,d}$ of simple types, where $\tau = \tau_{i_1,\dots,i_{e'}} = \sigma^{\phi^{i_1}} \otimes \cdots \otimes \sigma^{\phi^{i_{e'}}} \in \overline{\mathfrak{A}_{e'}^\times}$, $e' = m/f'$, $i_v \in \mathbb{Z}/(d,\ f)\mathbb{Z}$.

► **Weak Explicit Matching Theorem.** Let $\mathcal{S}_{\bar\chi}^A$ be the discrete series representations of $A^\times$ which contain $[\tau]_{\bar\chi,d}$, where $d = d(A)$. This is precisely one unramified twist class of discrete series representation. Then

$$JL(\mathcal{S}_{\bar\chi}^{A_1}) = \mathcal{S}_{\bar\chi}^{A_2}.$$

► **Central characters.** Let $\chi \in X_t(F_n^\times)$. It corresponds to $\bar\chi \in X(k_n^\times)$ by reduction.

Let $\mathcal{S}_\chi^A$ be those Π in $\mathcal{S}_{\bar\chi}^A$ with central character $\omega_\Pi = \chi|F^\times$. Then $\#\mathcal{S}_\chi^A = e = n/f$.

► **5. Extended types.**

The idea is that we want to extend τ (a simple level-0 type) on $\mathfrak{A}_{e'}^\times$ to a bigger group to single out one discrete series representation.

Main point. In general possible, but one has to change the order (also known to Morris).

We go from $\mathfrak{A}_{e'}$ to $\mathfrak{A}_r$, where $e' = (e,m)$, $e = n/f$, $r = m/(f,m)$. Thus $e'/r = (e,m)(f,m)/m$.

We have $\mathfrak{A}_r \supset \mathfrak{A}_{e'}$ and $\mathfrak{P}_r \subset \mathfrak{P}_{e'}$. Now let $(\Pi, V) \in \mathcal{S}_{\bar\chi}^A$. Let $V_r = V^{1+\mathfrak{P}_r}$. We have a Jacquet map (for finite fields) from V_r to $V_{e'}$. And this gives $Irr(V_r) \leftrightarrow Irr(V_{e'})/\sim$ for the irreducible components where certain equivalence relation is imposed on $Irr(V_{e'})$.

From above, we have the bijection $Irr(V_{e'}) \leftrightarrow (\mathbb{Z}/(d,\ f)\mathbb{Z})^{e'}$ and the equivalence relation can be made explicit. Moreover, the action of the normalizer can be transported to the set of equivalence classes. This gives a combinatorial result:

$$\langle t_r\rangle = N(\mathfrak{A}_r)/\mathfrak{A}_r^\times.$$

Proposition. All t_r-orbits in $Irr(V_r)$ have order a multiple of $c = f/(f,m)$, and there exists a unique orbit $\{\pi_1, \dots, \pi_c\}$ of order c. Moreover, π_v is parabolically induced from $\tau = \tau_{i_1,\dots,i_{e'}}$ for good choice of $i_1, \dots, i_{e'}$. Therefore, π_v is generic.

We have

$$\mathfrak{A}_r^\times \cdot F^\times \subset \mathfrak{A}_r^\times \langle t_r^c \rangle \subset N(\mathfrak{A}_r^\times).$$

These inclusions have indices $e = n/f$ and c respectively. We will do an extension in the first stage, and an induction in the second stage. For the first stage, there are e extensions, parameterized by $\chi_f \in X_t(F_f^\times)$ such that $\chi_f \circ N_{F_n/F_f} = \chi$.

Because π_v is generic if it has a Whittaker vector which allows to specify the extensions Σ_{χ_f,π_v} in such a way that $t_v \sum_{\chi_f,\pi_v} t_v^{-1} = \sum_{\chi_f,t_v\pi_v t_v^{-1}}$, We associate to χ_f the representation $\mathrm{Ind}(\Sigma_{\chi_f,\chi_v} = \tilde{\Sigma}_{\chi_f}$ of $N(\mathfrak{A}_r^\times)$. There is then a unique discrete series representation containing $\tilde{\Sigma}_{\chi_f}$.

▶ 6. Explicit matching.

We have $\Pi \in R_0^2(A^\times)$, from Langlands parameter $\Pi = \Pi^A_{\chi_f'}$, and from type $\Pi \supset \tilde{\Sigma}_{\chi_f}$. Both χ_f and χ_f' pull back to the same χ of $F_n^\times$. Call these constructions A and B.

The main result is to match these by computing the character Θ_π. via (A) (see Proposition 2) and (B):

Theorem. For well-chosen arguments $x \in \mathfrak{A}_r^\times t_r^c$, we get

$$\Theta_\pi(x) = \Theta_{\tilde{\Sigma}_{\chi_f}}(x) = \sum_{v=1}^{c} \Theta_{\Sigma_{\chi_f,\pi_v}}(x).$$

To andle the character $\Theta_{\Sigma_{\chi_f,\pi_v}}(x)$, we use that π_v is a $t := t_r^c$ invariant representation of $\overline{\mathfrak{A}_r^\times} = [\mathrm{GL}_s(k_D)]^r$ which has a well-defined Shintani descent $\rho_v = Sh_t(\pi_v)$, which is an irreducible representation of the fixed point group $(\overline{\mathfrak{A}_v^\times})^t \simeq \mathrm{GL}_s(k_c)$.

Then we are in a position to compare the character formulas via (A) and via (B) which gives the matching result:

$$\chi_f' = \omega_f^{m-(f,m)} \chi_f,$$

where ω_f is certain unramified character $F_f^\times$ of order 2.