

ON VECTOR BUNDLES DESTABILIZED BY FROBENIUS PULL-BACK

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§1. Introduction

Let X be an irreducible smooth projective curve of genus g over an algebraically closed field k of characteristic $p > 0$, and $F : X \rightarrow X$ the absolute Frobenius morphism on X . It is known that pulling back a stable vector bundle on X by F may destroy stability. One may measure the failure of (semi-)stability by the Harder-Narasimhan polygons of vector bundles.

In more formal language, let $n \geq 2$ be an integer, \mathcal{M} the coarse moduli space of stable vector bundles of rank n and a fixed degree on X . Applying a theorem of Shatz to the pull-back by F of the universal bundle (assuming the existence) on \mathcal{M} , we see that \mathcal{M} has a canonical stratification by Harder-Narasimhan polygons ([15]). We call this the *Frobenius stratification*. This interesting extra structure on \mathcal{M} is a feature of characteristic $p > 0$. However, very little is known about the strata of the Frobenius stratification. Scattered constructions of points outside of the largest (semi-stable) stratum can be found in [5], [16], [19]. Complete classification of such points is only known when $p = 2$, $n = 2$, and $g = 2$ by [10] and [13].

Our main result here settles the problem for the case of $p = 2$ and $n = 2$. On any curve X of genus ≥ 2 , we provide a complete classification of rank-2 semi-stable vector bundles V with F^*V not semi-stable. This also shows that the bound in [20, Theorem 3.1] is sharp. We also obtain fairly good information about the locus destabilized by Frobenius in the moduli space, including the irreducibility and the dimension of each non-empty Frobenius stratum. In particular we show that the locus of Frobenius destabilized bundles has dimension $3g - 4$ in the moduli space of semi-stable bundles of rank two. An interesting consequence of our classification is that high instability of F^*V implies high stability of V .

In addition, we show that the Gunning bundle descends when g is even. If g is odd, then the Gunning bundle twisted by any odd degree line bundle also descends.

We also construct stable bundles that are destabilized by Frobenius in the following situations: (1) $p = 2$ and $n = 4$, (2) $p = n = 3$, (3) $p = n = 5$ and $g \geq 3$.

The problem studied here can be cast in the generality of principal G -bundles over X , where G is a connected reductive group over k . More precisely, consider the pull-back by F of the universal object on the moduli stack of semi-stable principal G -bundles on X . Atiyah-Bott's generalization of the Harder-Narasimhan filtration should then give a canonical stratification of the moduli stack ([1], see also [4]).

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There is a connection between Frobenius destabilized bundles and (pre)-opers. The investigation of this connection is largely inspired by [2]. The new phenomena we observe here is that, in characteristic $p > 0$, pre-opers exist and they need not be opers (indeed all the examples of pre-opers provided here are not opers). When $p = 0$, all pre-opers are opers.

The results of this paper were discovered independently by the first two authors and the last two authors. When it was realized that there was a considerable overlap in the techniques and the results, we decided to write it jointly. The last two named authors thank C.-L. Chai for discussions and also wish to thank the hospitality of the National Center for Theoretical Sciences, Hsinchu, Taiwan.

§2. Generalities

(2.1) Notations The following notations are in force throughout this paper unless otherwise specified. Let X be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. Let Ω_X^1 be the sheaf of 1-forms on X and T_X the tangent bundle of X .

Let V be a vector bundle on X and denote by $F^*(V)$ the pull-back of V by F . If $V = L$ is a line bundle, then $F^*(L) = L^{\otimes p}$. We write $V^* = \text{Hom}_{\mathcal{O}_X}(V, \mathcal{O}_X)$ for the dual bundle of V . Denote by $\chi(V)$ the Euler characteristic of V . By Riemann-Roch,

$$\chi(E) = \deg(V) + \text{rank}(V)(1 - g).$$

Denote by $\mu(V) = \deg(V)/\text{rank}(V)$, the *slope* of V .

(2.2) Stability A vector bundle V is *stable* (resp. *semi-stable*) if for any non-zero sub-bundle $W \subset V$, $\mu(W) < \mu(V)$ (resp. $\mu(W) \leq \mu(V)$). A non-zero sub-bundle $W \subset V$ with $\mu(W) \geq \mu(V)$ will be called a *destabilizing sub-bundle*.

(2.3) Harder-Narasimhan filtration Let V be a vector bundle on X . Then there exists a unique filtration (see [15, 5.4]), called the *Harder-Narasimhan filtration*, by sub-bundles

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{h-1} \subset V_h = V$$

such that V_i/V_{i-1} is semi-stable of slope μ_i and

$$\mu_1 > \mu_2 > \cdots > \mu_h.$$

The data of $(\dim(V_i/V_{i-1}), \mu_i)$ can be encoded into a polygon, called the *Harder-Narasimhan polygon* (see [15, 11.1]). The Harder-Narasimhan polygon can be regarded as a measure of instability.

(2.4) A measure of stability Following [12], for a rank-2 vector bundle V , we put

$$s(V) = \deg(V) - 2 \max\{\deg(L) : L \hookrightarrow V\},$$

where the maximum is taken over all rank-1 sub-bundle of V . By definition, $s(V) > 0$ (resp. $s(V) \geq 0$) if and only if V is stable (resp. semi-stable). When $s(V) \leq 0$, the information of $(s(V), \deg(V))$ is the same as that of the Harder-Narasimhan polygon of V . Therefore, one may regard s as a measure of stability extrapolating the Harder-Narasimhan polygons, though it is only for the rank-2 case (for possible variants for the higher rank case, see [3]; for general reductive group, see [9]).

(2.5) For any vector bundle with a connection (V, ∇) , there exists a p -linear morphism of \mathcal{O}_X -modules, called the p -curvature of ∇ ,

$$\psi : T_X \rightarrow \text{End}(V)$$

which measures the obstruction to the Lie algebra homomorphism $\nabla : T_X \rightarrow \text{End}(V)$ being a homomorphism of p -Lie algebras. A connection is p -flat if ψ is zero. A vector bundle is p -flat if it admits a p -flat connection.

By a theorem of Cartier ([11, Theorem 5.1, page 190]), there exists a vector bundle W on X such that $F^*(W) \simeq V$ if and only if V carries a p -flat connection

$$\nabla : V \rightarrow \Omega_X^1 \otimes V$$

such that the natural map

$$F^*(V^{\nabla=0}) \rightarrow V,$$

(where $V^{\nabla=0}$ is the module of flat sections considered as an \mathcal{O}_X -module), is an isomorphism.

(2.6) Suppose (V, ∇) is a vector bundle with a connection and $W \subset V$ a sub-bundle. Then there is a natural map (the second fundamental form)

$$T_X \rightarrow \text{Hom}(W, V/W),$$

which is zero if and only if ∇ preserves W . By Cartier's theorem, if (V, ∇) is p -flat and $W \hookrightarrow V$ is a sub-bundle preserved by ∇ , then ∇ restricts to a p -flat connection on W .

(2.7) Let B_1 be the vector bundle defined by the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow F_*(\mathcal{O}_X) \rightarrow B_1 \rightarrow 0$$

The bundle B_1 is semi-stable of slope $g-1$ (and degree $(p-1)(g-1)$); moreover, for $p > 2$, $F^*(B_1)$ is not semi-stable [19]. For $p = 2$, B_1 is a theta characteristic, i.e. $B_1^{\otimes 2} = \Omega_X^1$ [19]. By [18, Proposition 1.1], B_1 is stable when $p = 3$ and $g \geq 2$.

(2.8) Lemma *Let L be a line bundle on X . Then*

$$\det(F_*L) = \det(B_1) \otimes L.$$

PROOF. See [8, Chapter 4, Exercise 2.6].

(2.9) Let V be a vector bundle on X . Then

$$\deg(F_*V) = \deg(V) + \text{rank}(V) \deg(B_1).$$

This follows from Riemann-Roch and the fact that $\chi(F_*V) = \chi(V)$ or by 2.8. In particular

$$\mu(F_*V) = \frac{1}{p}\mu(V) + (1 - \frac{1}{p})(g - 1).$$

(2.10) Duality Let V be a vector bundle on X . Following [17, section 1.16, page 70], we have

$$F_*(V)^* \simeq F_*(V^* \otimes (\Omega_X^1)^{\otimes(1-p)}).$$

Thus the dual of $F_*(V)$ is of the form $F_*(V')$. We will often make use of this fact together with the following simple lemma.

Lemma *Let V be a vector bundle of rank n on X , m an integer such that $0 < m < n$. The following are equivalent:*

- (i) *for all sub-bundle W of V of rank m , we have $\mu(W) < \mu(V)$ (resp. $\mu(W) \leq \mu(V)$);*
- (ii) *for all sub-bundle W' of V^* of rank $n - m$, we have $\mu(W') < \mu(V^*)$ (resp. $\mu(W') \leq \mu(V^*)$).*

§3. A general construction

(3.1) Proposition *Let V be a vector bundle on X . Then the adjunction map $F^*(F_*(V)) \rightarrow V$ is surjective and $\mu(F^*(F_*(V))) = \mu(V) + (p - 1)(g - 1) > \mu(V)$.*

PROOF. The surjectivity of the adjunction map is easily check by a local calculation. The formula for slope follows from 2.9. Hence $\mu(F^*(F_*(V))) > \mu(V)$.

Remark In 5.3, we prove a stronger assertion: $F^*(F_*(V))$ is highly unstable whenever V is semi-stable.

(3.2) Proposition *Let V be a semi-stable bundle on X .*

- (i) *For any rank-1 sub-bundle L of F_*V , we have*

$$\mu(L) \leq \mu(F_*V) - \frac{(p-1)(g-1)}{p}.$$

- (ii) *For any rank-2 sub-bundle E of F_*V , we have*

$$\mu(E) \leq \mu(F_*V) - \frac{1}{p} \left(\frac{pg}{2} - p - g + 1 \right).$$

PROOF. If $L \hookrightarrow F_*V$ is a line sub-bundle, then by adjunction, there is a non-zero morphism $F^*L \rightarrow V$. Therefore, $\mu(F^*L) \leq \mu(V)$; i.e., $p \cdot \mu(L) \leq \mu(V) = p \cdot \mu(F_*V) - (p-1)(g-1)$. Here we have made use of the formulas in 2.9. This proves (i).

Let $E \hookrightarrow F_*V$ be a sub-bundle of rank 2. Then by a theorem of Nagata [9], there is a line sub-bundle $L \hookrightarrow E$ such that $\mu(L) \geq \mu(E) - g/2$. Thus we have $\mu(E) \leq \mu(L) + g/2 \leq \mu(F_*V) - (p-1)(g-1)/p + g/2$. This proves (ii).

(3.3) Theorem (i) *Let $p = 2$ and V a stable bundle of rank two and even degree on X . Then $F_*(V)$ is a semi-stable bundle of rank 4 and $F^*(F_*(V))$ is not semi-stable.* (ii) *Suppose $p = 3$ (resp. $g \geq 3$ and $p = 5$). Let V be a line bundle on X . Then the bundle $F_*(V)$ is a stable bundle of rank 3 (resp. 5) and $F^*(F_*(V))$ is not semi-stable.*

PROOF. (i) By 2.10 and 3.1, it suffices to show that for any sub-bundle E of F_*V of rank ≤ 2 , we have $\mu(E) \leq \mu(F_*V)$. This is clear when $\text{rank } E = 1$ by 3.2 (i). Suppose that $\text{rank } E = 2$ and $\mu(E) > \mu(F_*V)$. The proof of 3.2 (ii) gives a line bundle $L \hookrightarrow E$ such that $\mu(E) \leq \mu(L) + g/2 \leq \mu(F_*V) + 1/2$.

The assumption that $\deg V$ is even implies that $\mu(F_*V) \in \frac{1}{2}\mathbb{Z}$. Thus we must have $\mu(E) = \mu(L) + g/2 = \mu(F_*V) + 1/2$. This gives $\mu(L) = \frac{1}{2}\mu(V)$ and $\mu(F^*L) = \mu(V)$, contradicting the stability of V as there is a non-zero morphism $F^*L \rightarrow V$ by adjunction.

(ii) By 2.10 and 3.1, it suffices to check that F_*V does not have a destabilizing sub-bundle of rank ≤ 1 (resp. ≤ 2). This is immediate from 3.2.

§4. A detailed study of the case of rank 2 and characteristic 2

Throughout this section, $p = 2$. We present our main results on the classification of rank-2 vector bundles destabilized by Frobenius, as well as the geometry of the Frobenius stratification.

(4.1) A result on the Gunning bundle We begin with an interesting observation about Gunning extensions, though this result is not needed in the sequel. Recall that B_1 is a theta-characteristic [19, §4]. The unique non-trivial extension $0 \rightarrow B_1 \rightarrow W \rightarrow B_1^{-1} \rightarrow 0$ is called *the Gunning extension* and the bundle W is called *the Gunning bundle*.

Proposition *Let ξ be a line bundle and $V = F_*(\xi \otimes B_1^{-1})$. The extension*

$$(*) \quad 0 \rightarrow \xi \otimes B_1 \rightarrow F^*V \rightarrow \xi \otimes B_1^{-1} \rightarrow 0$$

defines a class in $\text{Ext}^1(\xi \otimes B_1^{-1}, \xi \otimes B_1) \simeq H^1(X, B_1^2) \simeq k$. This class is trivial precisely when $\deg(\xi \otimes B_1^{-1})$ is even.

PROOF. Suppose that $\deg(\xi \otimes B_1^{-1})$ is even. Then we can write $L = \xi \otimes B_1^{-1} = M^2$. By [10, §2], there is an exact sequence $0 \rightarrow M \rightarrow V \rightarrow M \otimes B_1 \rightarrow 0$. Pulling back by F , we get $0 \rightarrow L \rightarrow F^*V \rightarrow L \otimes B_1^2 \rightarrow 0$. This shows that (*) is split.

Suppose that $L = \xi \otimes B_1^{-1}$ has odd degree $2n+1$. By a theorem of Nagata ([12], Cf. Remark 4.5), there is an exact sequence $0 \rightarrow M_1 \rightarrow V \rightarrow M_2 \rightarrow 0$, where M_1, M_2 are line

bundles with degrees n and $n + g$ respectively. From the exact sequence $0 \rightarrow M_1^2 \rightarrow F^*V \rightarrow M_2^2 \rightarrow 0$, we deduce that $\dim \text{Hom}(L, F^*V) \leq \dim \text{Hom}(L, M_1^2) + \dim \text{Hom}(L, M_2^2) = 0 + g = g$ by the Riemann-Roch formula. Since $\text{Hom}(L, \xi \otimes B_1) = H^0(X, B_1^2)$ has dimension g , any morphism $L \rightarrow F^*V$ factors through the sub-module $\xi \otimes B_1$ in (*). Therefore, (*) is not split.

Corollary *Let W be the Gunning bundle and ξ a line bundle of degree $\equiv g \pmod{2}$. Then there exists a stable bundle V such that $F^*V \simeq W \otimes \xi$. In particular, if g is even, then the Gunning bundle W is the Frobenius pull-back of a stable bundle.*

Remark This corollary is implicit in [13] in the case of an ordinary curve with $g = 2, p = 2$. In [5], Gieseker proved (by different methods) an analogous result in any characteristic when X is a Mumford curve.

(4.2) The basic construction Henceforth, fix an integer d . For an injection $V' \hookrightarrow V''$ of vector bundles of the same rank, define the *co-length* l of V' in V'' to be the length of the torsion \mathcal{O}_X -module V''/V' . Clearly, $s(V') \geq s(V'') - l$.

We now give a basic construction of stable vector bundles V of rank 2 with F^*V not semi-stable. Let $l \leq g - 2$ be a non-negative integer, L a line bundle of degree $d - 1 - (g - 2 - l)$, and V a sub-module of F_*L of co-length l , then $\deg V = d$ and $s(V) \geq (g - 1) - l > 0$ by 3.2. Therefore, V is stable.

On the other hand, by adjunction, there is a morphism $F^*V \rightarrow L$, and the kernel is a line bundle of degree $\geq d + 1 + (g - 2 - l) > d = \deg(F^*V)/2$. Therefore, F^*V is not semi-stable.

(4.3) Exhaustion Suppose that V is semi-stable of rank 2 and F^*V is not semi-stable.

Let $\xi = \det(V)$ and $d = \deg \xi = \deg V$. Since F^*V is not semi-stable and of degree $2d$, there are line bundles L, L' and an exact sequence $0 \rightarrow L' \rightarrow F^*V \rightarrow L \rightarrow 0$ with $\deg L' \geq d + 1$, $\deg L \leq d - 1$. By adjunction, this provides a non-zero morphism $V \rightarrow F_*L$. If the image is a line bundle M , we have $\deg M \geq d/2$ by semi-stability of V , and $\deg M \leq (d - 1 + g - 1)/2 - (g - 1)/2 = (d - 1)/2$ by 3.2. This is a contradiction.

Thus the image is of rank 2. Since $\deg V = d$ and $\deg(F_*L) \leq d + (g - 2)$, V is a sub-module of F_*L of co-length $l \leq g - 2$, and $\deg L = d - 1 - (g - 2 - l)$.

Thus the basic construction yields all semi-stable vector bundles V of rank 2, with F^*V not semi-stable.

(4.3.1) Corollary *If V is semi-stable of rank 2 with F^*V not semi-stable, then V is actually stable.*

(4.3.2) Corollary *The basic construction with $l = g - 2$ already yields all semi-stable vector bundles V of rank 2, with F^*V not semi-stable.*

PROOF. In fact, if $l < l' \leq g - 2$ and $L' = L \otimes \mathcal{O}(D)$ for some effective divisor D of degree $l' - l$ on X , then $V \hookrightarrow F_*L \hookrightarrow F_*L'$. Hence V is also a sub-module of F_*L' of co-length l' . Thus V arises from the basic construction with (l', L') playing the role of (l, L) .

(4.4) Classification Let L be a line bundle and let $Q = Q_l = Q_{l,L} = \text{Quot}_l(F_*L/X/k)$ be the scheme classifying sub-modules of F_*L of co-length l ([7, 3.2]). Let

$$\mathcal{V} \hookrightarrow \mathcal{O}_Q \boxtimes F_*L = (\text{id} \times F)_*(\mathcal{O}_Q \boxtimes L)$$

(sheaves on $Q \times X$) be the universal object on Q . By adjunction, we have a morphism $(1 \times F)^*\mathcal{V} \rightarrow \mathcal{O}_Q \boxtimes L$. Let \mathcal{F} be the cokernel. Then $\text{pr}_*\mathcal{F}$ is a coherent sheaf on Q , where $\text{pr} : Q \times X \rightarrow Q$ is the projection ([8, II.5.20]). By [8, III.12.7.2], the subset

$$\{q \in Q : \dim_{\kappa(q)}((\text{pr}_*\mathcal{F}) \otimes \kappa(q)) > 0\}$$

is closed. Its complement is an open sub-scheme, denoted by $Q^* = Q_l^* = Q_{l,L}^*$, of Q . Then Q^* parameterizes those V 's with surjective $F^*V \rightarrow L$.

Let $\overline{\mathcal{M}}$ be the coarse moduli space of rank-2 semi-stable vector bundles of degree d on X . Let \mathcal{M} be the open sub-scheme parameterizing stable vector bundles and $\mathcal{M}_1(k) \subset \overline{\mathcal{M}}(k)$ the subset of those V 's such that F^*V is not semi-stable. By 4.3.1, $\mathcal{M}_1(k) \subset \mathcal{M}(k)$.

Proposition *The basic construction gives a bijection*

$$\coprod_{\substack{0 \leq l \leq g-2 \\ \deg L = d-1-(g-2-l)}} Q_{l,L}^*(k) \rightarrow \mathcal{M}_1(k),$$

where the disjoint union is taken over all $l \in [0, g - 2]$ and a set of representatives of all isomorphism classes of line bundles L of degree $d - 1 - (g - 2 - l)$.

PROOF. By 4.3, the map is a surjection. Now suppose that $(l, L, V \subset F_*L)$ and $(l', L', V' \subset F_*L')$ give the same point in $\mathcal{M}_1(k)$, i.e. $V \simeq V'$. Since the unstable bundle F^*V has a unique quotient line bundle of degree $< \deg(V)/2$ (i.e. the second graded piece of the Harder-Narasimhan filtration), which is isomorphic to L , we must have $L = L'$. Consider the diagram

$$\begin{array}{ccc} F^*V & \longrightarrow & L \\ \wr \downarrow & & \parallel \\ F^*V' & \longrightarrow & L', \end{array}$$

where the vertical arrow is induced from an isomorphism $V \xrightarrow{\sim} V'$ and the horizontal arrows are the unique quotient maps. This diagram is commutative up to a multiplicative scalar in k^* . By adjunction, $V \hookrightarrow F_*L$ and $V' \hookrightarrow F_*L$ have the same image. In other words, $V = V'$ as sub-modules of F_*L . This proves the injectivity of the map.

(4.5) Frobenius Stratification To ease the notation, let $d_l = d - 1 - (g - 2 - l)$. Let $\text{Pic}^{d_l} X$ be the moduli space of line bundles of degree d_l on X , and $\mathcal{L} \rightarrow \text{Pic}^{d_l}(X) \times X$ the universal line bundle.

By [7, 3.2], there is a scheme $\mathcal{Q} = \mathcal{Q}_l = \text{Quot}_l((\text{id} \times F)_*\mathcal{L}/(\text{Pic}^{d_l}(X) \times X)/\text{Pic}^{d_l} X) \xrightarrow{\pi} \text{Pic}^{d_l} X$ such that \mathcal{Q}_x (the fiber at x) is $Q_{\mathcal{L}_x}$ for all $x \in (\text{Pic}^{d_l} X)(k)$. By the same argument as before, there is an open sub-scheme $\mathcal{Q}^* \subset \mathcal{Q}$ such that $\mathcal{Q}_x^* = Q_{\mathcal{L}_x}^*$ for all $x \in \text{Pic}^{d_l}(X)(k)$. The scheme \mathcal{Q} is projective over $\text{Pic}^{d_l}(X)$ ([7, 3.2]), hence is proper over k . By checking the condition of formal smoothness (cf. [15, 8.2.1]), it can be shown that \mathcal{Q} is smooth over $\text{Pic}^{d_l}(X)$, hence is smooth over k .

The Frobenius stratification on the coarse moduli scheme \mathcal{M} is defined canonically using Harder-Narasimhan polygons of Frobenius pull-backs. Concretely, for $j \geq 0$, let P_j be the polygon from $(0, 0)$ to $(1, d + j)$ to $(0, 2d)$. Let $\mathcal{M}_0 = \overline{\mathcal{M}}$, and for $j \geq 1$, let $\mathcal{M}_j(k)$ be the subset of $\overline{\mathcal{M}}(k)$ parameterizing those V 's such that the Harder-Narasimhan polygons ([15, 11.1]) of F^*V lie above or are equal to P_j . Notice that $\mathcal{M}_1(k)$ agrees with the one defined in 4.4.

As mentioned in the introduction, the existence of a universal bundle on \mathcal{M} would imply that each $\mathcal{M}_j(k)$ is Zariski closed by Shatz's theorem [15, 11.1, last remark]. In general, one can show that $\mathcal{M}_j(k)$ is closed by examining the GIT (geometric invariant theory) construction of $\overline{\mathcal{M}}$. This fact also follows from our basic construction:

Theorem *The subset $\mathcal{M}_j(k)$ is Zariski closed in $\overline{\mathcal{M}}(k)$, hence underlies a reduced closed sub-scheme \mathcal{M}_j of $\overline{\mathcal{M}}$. The scheme \mathcal{M}_j is proper. The Frobenius stratum $\mathcal{M}_j \setminus \mathcal{M}_{j+1}$ is non-empty precisely when $0 \leq j \leq g - 1$. For $1 \leq j \leq g - 1$, write $l = g - 1 - j$. Then there is a canonical morphism*

$$\mathcal{Q}_l \rightarrow \overline{\mathcal{M}}$$

which has scheme-theoretic image \mathcal{M}_j and induces a bijection from $\mathcal{Q}_l^(k)$ to $\mathcal{M}_j(k) \setminus \mathcal{M}_{j+1}(k)$.*

PROOF. Suppose $0 \leq l \leq g - 2$ and $j + l = g - 1$. The universal object $\mathcal{V} \rightarrow \mathcal{Q}_l \times X$ is a family of stable vector bundles on X . This induces a canonical morphism $\mathcal{Q}_l \rightarrow \overline{\mathcal{M}}$. The image of $\mathcal{Q}_l(k)$ is precisely $\mathcal{M}_j(k)$ by (the proof of) 4.3.2. Since \mathcal{Q}_l is proper, \mathcal{M}_j is proper and closed in $\overline{\mathcal{M}}$. The rest of the proposition follows from 4.4 and 4.3, and the fact that $\mathcal{Q}_l^*(k)$ is non-empty for $0 \leq l \leq g - 2$ (see 4.6.3).

Remark By a theorem of Nagata ([12], [9]), $s(V) \leq g$ for all V . Therefore, $s(V) \leq g$ if $\deg V \equiv g \pmod{2}$, and $s(V) \leq g - 1$ if $\deg V \not\equiv g \pmod{2}$. By 3.2, $V = F_*L$ achieves the maximum value of s among rank-2 vector bundles of the same degree.

By the preceding theorem, vector bundles of the form $V = F_*L$ are precisely members of the smallest non-empty Harder-Narasimhan stratum \mathcal{M}_{g-1} . Therefore, in a sense V is most stable yet F^*V is most unstable. More generally, for $1 \leq j \leq g - 1$, we have (from

4.2)

$$s(\mathcal{M}_j(k)) \geq \begin{cases} j & \text{if } d \equiv j \pmod{2}, \\ j+1 & \text{if } d \not\equiv j \pmod{2}. \end{cases}$$

Therefore, high instability of F^*V implies high stability of V .

(4.6) Irreducibility We will make use of the following simple lemma.

(4.6.1) Lemma *Let Y be a proper scheme over k , S an irreducible scheme of finite type over k of dimension s , r an integer ≥ 0 , and $f : Y \rightarrow S$ a surjective morphism. Suppose that all fibers of f are irreducible of dimension r . Then Y is irreducible of dimension $s+r$.*

(4.6.2) Lemma *The scheme $\mathcal{Q} = \mathcal{Q}_l$ is irreducible of dimension $2l+g$.*

PROOF. There is a surjective morphism ([7, §6])

$$\delta : \mathcal{Q} \rightarrow \text{Div}^l(X) = \text{Sym}^l(X), \quad q \mapsto \sum_{P \in X(k)} \text{length}_{\mathcal{O}_P}((F_*\mathcal{L}_{\pi(q)})/\mathcal{V}_q) \cdot P.$$

The morphism $\mathcal{Q} \rightarrow \text{Div}^l(X) \times \text{Pic}^{d_l}(X)$ is again a surjection. The fibers are irreducible schemes of dimension l according to the last lemma of [14]. Since \mathcal{Q} is proper, the result follows from 4.6.1.

(4.6.3) Lemma *\mathcal{Q}^* is open and dense in \mathcal{Q} .*

PROOF. By the construction in 4.4 and 4.5, \mathcal{Q}^* is open in \mathcal{Q} . Since \mathcal{Q} is irreducible of dimension $2l+g$, it suffices to show that \mathcal{Q}^* is non-empty. We will do more by exhibiting an open subset of \mathcal{Q}^* of dimension $2l+g$.

Indeed, let $B(X, l) \subset \text{Div}^l(X)$ be the open sub-scheme parameterizing multiplicity-free divisors of degree l , also known as the configuration space of unordered l points in X . Let U be the inverse image of $B(X, l) \times \text{Pic}^{d_l}(X)$ under $\mathcal{Q}^* \rightarrow \text{Div}^l(X) \times \text{Pic}^{d_l}(X)$. A quick calculation shows that each fiber of $U \rightarrow B(X, l) \times \text{Pic}^{d_l}(X)$ is isomorphic to \mathbb{A}^l . Therefore, U is an open subset of \mathcal{Q}^* of dimension $2l+g$.

(4.6.4) Theorem *For $1 \leq j \leq g-1$, \mathcal{M}_j is proper, irreducible, and of dimension $g+2(g-1-j)$. In particular, \mathcal{M}_1 is irreducible and of dimension $3g-4$.*

(4.7) Fixing the determinant Fix a line bundle ξ of degree d . Let $\overline{\mathcal{M}}(\xi) \subset \overline{\mathcal{M}}$ be the closed sub-scheme of $\overline{\mathcal{M}}$ parameterizing those V 's with $\det(V) = \xi$. Let $\mathcal{M}_j(\xi) = \overline{\mathcal{M}}(\xi) \cap \mathcal{M}_j$ for $j \geq 0$.

Remark For $1 \leq j \leq g-1$, $\dim \mathcal{M}_j(\xi) = 2(g-1-j)$. In particular, $\dim \mathcal{M}_1(\xi) = 2(g-2)$.

PROOF. Since $\mathcal{M}_j(\xi)$ is nothing but the fiber of the surjective morphism $\det : \mathcal{M}_j \rightarrow \text{Pic}^d(X)$, it has dimension $2(g-1-j)$ for a dense open set of $\xi \in \text{Pic}^d(X)(k)$. However, $\mathcal{M}_j(\xi_1)$ is isomorphic to $\mathcal{M}_j(\xi_2)$ for all $\xi_1, \xi_2 \in \text{Pic}^d(X)(k)$, via $V \mapsto V \otimes L$, where $L^2 \simeq \xi_2 \otimes \xi_1^{-1}$. Thus the remark is clear.

A slight variation of the above argument shows that $\mathcal{M}_j(\xi)$ is irreducible. Alternatively, assume $1 \leq j \leq g-1$. Let $l = g-1-j$ and let $\mathcal{Q}(\xi) = \mathcal{Q}_l(\xi)$ be the inverse image of ξ under $\mathcal{Q} \rightarrow \text{Pic}^d(X)$, $q \mapsto \det(\mathcal{V}_q)$. Since $\det(\mathcal{V}_q) = B_1 \otimes \mathcal{L}_{\pi(q)} \otimes \mathcal{O}(-\delta(q))$, the morphism $\det : \mathcal{Q} \rightarrow \text{Pic}^d(X)$ factors as

$$\mathcal{Q} \rightarrow \text{Div}^l(X) \times \text{Pic}^{d_l}(X) \xrightarrow{\psi} \text{Pic}^d(X),$$

where ψ is $(D, L) \mapsto B_1 \otimes L \otimes \mathcal{O}(-D)$. It is clear that $\psi^{-1}(\xi)$ is isomorphic to $\text{Div}^l(X)$, and hence is an irreducible variety.

The fibers of $\mathcal{Q}(\xi) \rightarrow \psi^{-1}(\xi)$ are just some fibers of $\mathcal{Q} \rightarrow \text{Div}^l(X) \times \text{Pic}^{d_l}(X)$; hence they are irreducible of dimension l as in the proof of 4.6.2. Being a closed sub-scheme of \mathcal{Q} , $\mathcal{Q}(\xi)$ is proper, thus, irreducible by 4.6.1. Now it is easy to deduce

Theorem *There is a canonical (Frobenius) stratification by Harder-Narasimhan polygons*

$$\emptyset = \mathcal{M}_g(\xi) \subset \mathcal{M}_{g-1}(\xi) \subset \cdots \subset \mathcal{M}_0(\xi) = \overline{\mathcal{M}}(\xi),$$

with $\mathcal{M}_j(\xi)$ non-empty, proper, irreducible, and of dimension $2(g-1-j)$ for $1 \leq j \leq g-1$.

(4.8) A variant Let $\mathcal{M}'(k)$ be the subset of $\overline{\mathcal{M}}(k)$ consisting of those V such that F^*V is not stable. Clearly, $\mathcal{M}'(k) \supset \mathcal{M}_1(k)$.

By 4.3.1, the closed subset $\mathcal{M}^{\text{ns}}(k) = \overline{\mathcal{M}}(k) \setminus \mathcal{M}(k)$ is contained in $\mathcal{M}'(k) \setminus \mathcal{M}_1(k)$. On the other hand, if $V \in \mathcal{M}'(k) \setminus \mathcal{M}^{\text{ns}}(k)$, the argument of 4.3 shows that there is a line bundle L of degree d such that $V \hookrightarrow F_*L$ is a sub-module of co-length $\leq g-1$. Conversely, the argument of 4.2 shows that if V is of co-length $\leq g-1$ in F_*L for some L of degree d , then $V \in \mathcal{M}'(k)$.

Thus we conclude that $\mathcal{M}'(k)$ is the union of $\mathcal{M}^{\text{ns}}(k)$ and the image $\mathcal{M}'_0(k)$ of $\mathcal{Q}_{g-1}(k)$ for a suitable morphism $\mathcal{Q}_{g-1} \rightarrow \overline{\mathcal{M}}$, where \mathcal{Q}_{g-1} is defined in 4.5. It follows that $\mathcal{M}'_0(k)$ and $\mathcal{M}'(k)$ are Zariski closed in $\overline{\mathcal{M}}(k)$, hence are sets of k -points of reduced closed sub-scheme \mathcal{M}'_0 and \mathcal{M}' of $\overline{\mathcal{M}}$.

Theorem *The scheme \mathcal{M}'_0 is irreducible of dimension $3g-2$. It contains two disjoint closed subsets: $\mathcal{M}'_0 \cap \mathcal{M}^{\text{ns}}$, which is irreducible of dimension $2g-1$ when d is even and empty when d is odd, and \mathcal{M}_1 , which is irreducible of dimension $3g-4$.*

Remark $\mathcal{M}' \setminus \mathcal{M}_1$ is the first stratum in the s -stratification ([12]) which is not a Harder-Narasimhan stratum. The other s -strata are more complicated and not pursued here.

PROOF. Since \mathcal{Q}_{g-1} is irreducible, \mathcal{M}'_0 is irreducible. We now analyze $\mathcal{M}'_0 \cap \mathcal{M}^{\text{ns}}$. Suppose that $V \in \mathcal{M}'_0(k) \cap \mathcal{M}^{\text{ns}}(k)$. Then $d = \deg V$ is even and there exists L of degree d such that V is a sub-module of F_*L of co-length $g - 1$. By assumption, there is a sub-bundle M of V of degree $d/2$. Adjunction applied to the composition $M \hookrightarrow V \hookrightarrow F_*L$ provides a non-zero morphism $F^*(M) = M^2 \rightarrow L$. This implies that $M^2 \simeq L$. We may assume that $L = M^2$. Since there is only one (modulo k^*) non-zero morphism $M^2 \rightarrow L$, there is only one non-zero morphism $M \rightarrow F_*(F^*M)$. By [10, §2], this morphism is part of an exact sequence $0 \rightarrow M \rightarrow F_*(F^*M) \rightarrow M \otimes B_1 \rightarrow 0$. Thus to have V is to have a sub-module of $M \otimes B_1$ of co-length $g - 1$. Conversely, starting with a sub-module of $M \otimes B_1$ of co-length $g - 1$, we obtain a vector bundle $V \in \mathcal{M}'_0(k) \cap \mathcal{M}^{\text{ns}}(k)$ as the inverse image of that sub-module in $F_*(F^*M)$.

The sub-modules of $M \otimes B_1$ of co-length $g - 1$ are of the form $M \otimes B_1 \otimes \mathcal{O}(-D)$ for $D \in \text{Div}^{g-1}(X)(k)$. Thus there is a morphism $\pi' : \mathcal{Q}' = \text{Div}^{g-1}(X) \times \text{Pic}^{d/2}(X) \rightarrow \overline{\mathcal{M}}$ inducing a surjection $\mathcal{Q}'(k) \rightarrow \mathcal{M}'_0(k) \cap \mathcal{M}^{\text{ns}}(k)$. We claim that this morphism is generically finite of separable degree at most 2. This claim implies that $\mathcal{M}'_0 \cap \mathcal{M}^{\text{ns}}$ is irreducible of dimension $2g - 1$.

Indeed, there is an open subset U of $\text{Div}^{g-1}(X)(k)$ such that if $D, D' \in U$ are distinct, then $D \not\sim D'$. We now show that $\pi'|_U(U \times \text{Pic}^{d/2}(X)(k))$ is at most 2-to-1. Suppose that $D \in U$, $M \in \text{Pic}^{d/2}(X)(k)$, and $\pi'(D, M) = V$. Then V has at most two isomorphism classes of rank-1 sub-bundles of degree $d/2$, and M is one of them. After obtaining M , one can determine D uniquely by the condition $\det(V) \simeq M^2 \otimes B_1 \otimes \mathcal{O}(-D)$. This proves the claim.

Next, we consider the morphism $\mathcal{Q}_{g-1} \rightarrow \mathcal{M}'_0$. It induces a surjection $\mathcal{Q}_{g-1}^*(k) \rightarrow \mathcal{M}'_0(k) \setminus \mathcal{M}_1(k)$. Again the claim is that the morphism is generically finite of separable degree at most 2. This claim implies that \mathcal{M}'_0 is irreducible of dimension $3g - 2$.

Indeed, let U be the open subset of $\mathcal{Q}_{g-1}^*(k)$ consisting of those q 's such that $\mathcal{O}(2\delta(q)) \not\cong \Omega_{X/k}$. Now assume that $q \in U$ gives rise to $V \in \mathcal{M}'_0(k)$. Then there is an exact sequence $0 \rightarrow L \otimes B_1^2 \otimes \mathcal{O}(-2\delta(q)) \rightarrow F^*V \rightarrow L \rightarrow 0$, where $L = \mathcal{L}_{\pi(q)}$. The assumption on q implies that F^*V has at most 2 quotient line bundles of degree d , say $F^*V \rightarrow L_1$ and $F^*V \rightarrow L_2$. Then q must be one of the two data $V \hookrightarrow F_*L_1$ or $V \hookrightarrow F_*L_2$ provided by adjunction. This proves the claim.

(4.9) Example When $g = 2$, $\mathcal{M}_1(\xi)$ is a single point, corresponding to the vector bundle $F_*(\xi \otimes B_1^{-1})$.

When $\xi = B_1$, this refines a result of [10, 1.1], which says that $\mathcal{M}_1(\xi)$ is a single $\text{Pic}(X)[2]$ -orbit.

When $\xi = \mathcal{O}_X$, this extends a theorem of Mehta [10, 3.2], which states that there are only finitely many rank-2 semi-stable vector bundles V 's on X with $\det(V) = \mathcal{O}_X$ and F^*V not semi-stable when $p \geq 3, g = 2$. We now have this result for $p = 2, g = 2$ with the

stronger conclusion of uniqueness.

§5. Pre-operators and operators

This section is largely inspired by the work Beilinson and Drinfel'd [2]. We show that pre-operators with connections of p -curvature zero provide, under additional assumptions, examples of Frobenius destabilized bundles. In small characteristics we describe the lowest Frobenius stratum in terms of pre-operators.

(5.1) Pre-operators Let V be a vector bundle on X with a flat connection ∇ . Suppose that $\{V_i\}_{0 \leq i \leq l} \subset V$ is an increasing filtration by sub-bundles such that

1. $V_0 = 0, V_l = V,$
2. $\nabla(V_i) \subset V_{i+1} \otimes \Omega_X^1$ for $0 \leq i \leq l-1,$
3. $V_i/V_{i-1} \xrightarrow{\nabla} (V_{i+1}/V_i) \otimes \Omega_X^1$ is an isomorphism for $1 \leq i \leq l-1.$

Then $(V, \nabla, \{V_i\})$ is said to be a *pre-oper*. A pre-oper is *p -flat* if ∇ has p -curvature zero.

Remark Let $(V, \nabla, \{V_i\}_{0 \leq i \leq l})$ be a pre-oper. If $g \geq 2$ and V_1/V_0 is semi-stable, then the filtration $\{V_i\}_{0 \leq i \leq l}$ is nothing but the Harder-Narasimhan filtration of V .

(5.2) Operators Let $\mathcal{D}_X = \text{Diff}(\mathcal{O}_X, \mathcal{O}_X)$ be the ring of differential operators on X . An *oper* $(V, \nabla, \{V_i\}_{0 \leq i \leq l})$ is a pre-oper such that the connection ∇ on V extends to a structure of \mathcal{D}_X -module on V .

Remark (i) By a Theorem of Katz (see [6, Theorem 1.3, page 4]), V is \mathcal{D}_X -module if and only if there exists a sequence of vector bundles V^i such that $F^*(V^1) = V$ and $F^*(V^{i+1}) = V^i$. In particular, if (V, ∇) is a vector bundle such that ∇ extends to a \mathcal{D}_X -module structure on V then ∇ has p -curvature zero, so any oper is automatically p -flat.

(ii) When $l = \text{rank } V$, what we called an oper here is the same as an GL_l -oper as defined in [2].

(5.3) A canonical filtration Let W be a vector bundle on X . We define a canonical increasing filtration on $V = F^*(F_*W)$ by abelian sub-sheaves $\{V_i\}_{0 \leq i \leq p}$, as follows:

$$\begin{aligned} V_p &= V, \\ V_{p-1} &= \ker(V_p = F^*(F_*W) \rightarrow W), \\ V_i &= \ker\left(V_{i+1} \xrightarrow{\nabla^{\text{Cartier}}} V \otimes \Omega_X^1 \rightarrow (V/V_{i+1}) \otimes \Omega_X^1\right), \quad 0 \leq i \leq p-2. \end{aligned}$$

It is elementary to check by induction that each V_i is actually an \mathcal{O}_X -sub-module of V .

Theorem (i) V_p/V_{p-1} is isomorphic to W .

- (ii) $(V, \nabla^{\text{Cartier}}, \{V_i\}_{0 \leq i \leq p})$ is a pre-oper.
- (iii) If $g \geq 2$ and W is semi-stable, $\{V_i\}_{0 \leq i \leq p}$ is simply the Harder-Narasimhan filtration on V .

PROOF. It is clear that (i) and (ii) imply (iii). To prove (i) and (ii), we notice that the definition of pre-opers and the formation of the filtration $\{V_i\}_{0 \leq i \leq p}$ can be made on any smooth 1-dimensional noetherian scheme over k , and statements (i) and (ii) make sense in this context. In fact, the statements being local, we are reduced to the case of a free \mathcal{O}_X -module W . Moreover, all the relevant formations commute with direct sums, hence we are reduced to the case of $W = \mathcal{O}_X$.

We can even reduce to the case $X = \text{Spec } k[[t]]$ and use an explicit calculation to complete the proof. Alternatively, one can check that the construction of [19, Remarkques 4.1.2 (2)] gives the same filtration and proves the theorem.

Remark If L is a line bundle such that $V = F_*L$ is stable, then the above result shows that F^*V is highly unstable, and it is likely that V is in a minimal Frobenius stratum (this is indeed the case in characteristic two). At least, the bound in [20, Theorem 3.1, page 51] is reached by V when the rank is p .

We refer to Theorem 3.3 (ii) for conditions ensuring stability of F_*L . The following is a partial converse to the above theorem.

Proposition *Assume that F_*L is stable for any line bundle L . Let $(V, \nabla, \{V_i\}_{0 \leq i \leq p})$ be a p -flat pre-oper with $\text{rank}(V) = p$ and $V^{\nabla=0}$ stable. Then $V^{\nabla=0} \simeq F_*(L)$ for a suitable line bundle L on X .*

PROOF. As $0 \subset V_1 \subset \dots \subset V_{p-1} \subset V_p = V$ is a pre-oper of rank p , $V/V_{p-1} = L$ is a line bundle. The morphism $F^*(V^{\nabla=0}) = V \rightarrow V/V_{p-1} = L$ gives by adjunction a non-zero morphism $V^{\nabla=0} \rightarrow F_*(L)$. Since these bundles are stable and of the same degree, the map $V^{\nabla=0} \rightarrow F_*(L)$ is an isomorphism.

(5.4) The underlying bundle of a pre-oper is typically unstable. In some circumstances, the Frobenius descent of a p -flat pre-oper is (semi)-stable. This provides a way of constructing Frobenius destabilized bundles in terms of pre-opers.

Proposition *Let $(V, \nabla, \{V_i\}_{0 \leq i \leq l})$ be a p -flat pre-oper with V_1 semi-stable of rank r_1 .*

- (i) Suppose $p > l^2(l-1)(g-1)r_1$. Then $V^{\nabla=0}$ is semi-stable.
- (ii) Suppose $r_1 = 1$ and $p > l^2(l-1)(g-1)$. Then $V^{\nabla=0}$ is stable.

PROOF. (i) A direct computation shows that $\mu(V_1) = \mu(V) + (l-1)(g-1)/r_1$. Since V_1 is semi-stable, $\{V_i\}_{0 \leq i \leq l}$ is the Harder-Narasimhan filtration of V . In particular, if $W \subset V$, then $\mu(W) \leq \mu(V_1)$.

Let $V' = V^{\nabla=0}$. Then $F^*(V') = V$. Suppose that V' is not semi-stable, and $W' \subset V'$ is such that $\mu(W') > \mu(V')$. Then $W = F^*(W')$ satisfies $\mu(W) \leq \mu(V_1)$, and

$$\frac{\mu(V)}{p} = \mu(V') < \mu(W') = \frac{\mu(W)}{p} \leq \frac{\mu(V_1)}{p} = \frac{\mu(V)}{p} + \frac{(l-1)(g-1)}{p \cdot r_1}.$$

However, $\mu(W')$ and $\mu(V')$ are fractions of the form a/b , with $a, b \in \mathbb{Z}$, $0 < b \leq l \cdot r_1$. Therefore, $\mu(W') - \mu(V') \geq (l \cdot r_1)^{-2}$. This contradicts the assumption on p .

(ii) Let $V' = V^{\nabla=0}$ and $0 = W'_0 \subset \cdots \subset W'_s = V'$ be a Jordan-Hölder series for V' . Then each W'_i/W'_{i-1} is stable of slope μ/p , where $\mu = \mu(V)$. Let $W_i = F^*(W'_i/W'_{i-1})$, $\mu_{\max}(W_i)$ be the largest possible slope of sub-bundles of W_i and $\mu_{\min}(W_i)$ be the smallest possible slope of quotient-bundles of W_i . By definition, $\mu_{\min}(W_i) \leq \mu \leq \mu_{\max}(W_i)$.

Let i_0 be the smallest integer such that $F^*(W'_{i_0}) \rightarrow V_i/V_{i-1}$ is non-zero. Then $\mu_{\min}(W_{i_0}) \leq \mu(V_i/V_{i-1}) = \mu - (l-1)(g-1)$. Similarly, there exists an index i_1 such that $\mu_{\max}(W_{i_1}) \geq \mu(V_1) = \mu + (l-1)(g-1)$.

A theorem of Sun [20, Theorem 3.1] asserts that

$$\mu_{\max}(W_i) - \mu_{\min}(W_i) \leq (\text{rank}(W_i) - 1)(2g - 2).$$

This implies that $\text{rank}(W_{i_0}) \geq (l+1)/2$ and $\text{rank}(W_{i_1}) \geq (l+1)/2$. Thus $i_0 = i_1$ and

$$(l-1)(2g-2) \leq \mu_{\max}(W_{i_0}) - \mu_{\min}(W_{i_0}) \leq (\text{rank}(W_{i_0}) - 1)(2g-2)$$

by Sun's theorem again. Therefore, $\text{rank } W_{i_0} \geq l$ and this forces W'_{i_0} to be the only Jordan-Hölder factor of V' .

Remark The bound on p can often be improved for particular $(l, g, r_1, \text{deg}(V_1))$. This is clear from the proof.

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