SMOOTH MODELS ASSOCIATED TO CONCAVE FUNCTIONS IN BRUHAT-TITS THEORY

JIU-KANG YU[1]

VERSION 1.3, DECEMBER 13, 2002

§0 Introduction

Let $\mathcal{O}$ be a henselian discrete valuation ring with perfect residue field, G a connected reductive group over k, the quotient field of $\mathcal{O}$. Let S be a maximal k-split torus of G, $\Phi = \Phi(G, S)$ the corresponding root system. For simplicity, in this introduction we assume that $\Phi(G, S)$ is reduced.

Fix a point x on the apartment $A(G, S)$ attached to S. According to Bruhat-Tits theory, x determines a filtration $\{U_a(k)_{x,r}\}_{r\in\mathbb{R}}$ for each root subgroup U_a of G. Let $f : \Phi \cup \{0\} \to \mathbb{R}$ be a concave function (see 8.2 for the definition) with $f(0) = 0$, and let $G(k)_{x,f}$ be the subgroup generated by $U_a(k)_{x,f(a)}$ and the maximal bounded subgroup of $S(k)$. Bruhat-Tits proved the following fundamental result:

***0.1* Theorem** *There is a canonical affine smooth group scheme $\underline{G}_{x,f}$ over $\mathcal{O}$, with generic fiber G, such that*

(i) $\underline{G}_{x,f}(\mathcal{O}) = G(k)_{x,f}$;

(ii) *for each $a \in \Phi$, the schematic closure $\underline{U}_a$ of U_a in $\underline{G}_{x,f}$ is smooth;*

(iii) *for any system Φ^+ of positive roots in Φ, the multiplication morphism*

$$\Big(\prod_{a\in\Phi^-} \underline{U}_a\Big) \times \underline{S} \times \Big(\prod_{a\in\Phi^+} \underline{U}_a\Big) \to \underline{G}_{x,f},$$

is an open immersion, where $\underline{S}/\mathcal{O}$ is the Néron-Raynaud model of S/k, and the two products $\prod_{a\in\Phi^\pm}$ can be taken in any order.

For example, by taking $f(a) = 0$ for all $a \in \Phi \cup \{0\}$, we get the group scheme canonically associated to the parahoric subgroup $G(k)_{x,0}$ of $G(k)$.

However, the assumption $f(0) = 0$ is quite restrictive in our view, e.g. it hinders one from discussing natural results concerning congruence subgroups in this context (see 8.8). Two possible reasons for this restriction are as follows. Firstly, Bruhat-Tits did not put a filtration on $S(k)$ (which can be thought as the "root subgroup" $U_0(k)$ attached to $a = 0$), and hence they can only discuss results using $\underline{S}(\mathcal{O})$, which should be considered as the level zero filtration group of $U_0(k)$.

---

[1] partially supported a Sloan Foundation Fellowship and by grant DMS 0100678 from the National Science Foundation

Secondly, the assumption $f(0) = 0$ also seems to come from an intrinsic limitation of the framework of Bruhat-Tits to construct the associated group schemes from their theory of schematic root datum (see [BT2, 3.3.1 (DRS 0)]).

Moy-Prasad and Schneider-Stuhler have defined a filtration on $S(k)$ (more generally on the centralizer of S in $G(k)$ when G is not split), making it possible to consider $G(k)_{x,f}$ for arbitrary concave f. In particular, the constant function f with $f(a) = r$ for all $a \in \Phi \cup \{0\}$ gives the Moy-Prasad group $G(k)_{x,r}$, which has been extremely useful in the representation theory of p-adic groups.

0.2 The purpose of this paper is three-fold. First, we extend Theorem 0.1 to the case of an arbitrary concave function f (removing the hypothesis $f(0) = 0$). In particular, we construct canonical smooth models associated to Moy-Prasad groups and Schneider-Stuhler groups.

Our result provides a link between the Moy-Prasad filtrations on a p-adic group and those on its Lie algebra. In addition, it should facilitate employing machinery of algebraic geometry into representation theory. The reader is also referred to the upcoming work of Aubert and Cunningham for possible applications.

0.3 The second purpose of this paper is to offer a treatment of Theorem 0.1 with minimal dependence on [BT2]. For most applications of Bruhat-Tits theory (e.g. to number theory and representation theory), the current paper may replace a large part of [BT2]. Our approach is completely different from that of [BT2], and is much simpler. It is also different from the approach using the Artin-Weil theorem, hinted in [BT2, 3.1.7] and carried out in [L]. The main idea is to use the theory of dilatation and group smoothening systematically [BLR].

To be more precise, we now recall the organization of the monumental work of Bruhat and Tits [BT1], [BT2]. Chapter 1 ([BT1]) is written in the language of abstract group theory. There, the notion of a valuation ([BT1, 6.2]) of a group $\mathcal{G}$ with root datum ([BT1, 6.1]) is defined and explored. Whenever we have such a valuation, there is an associated affine building, and actually every point on the building gives rise to a valuation of root datum. We can then derive a lot of geometric/algebraic structures: the apartments, the polysimplicial structure, the double Tits system, the parahoric subgroups, the bounded subgroups associated to concave functions, and so on.

In chapter 2, it is shown that the theory of chapter 1 applies to a connected reductive group G over k. That is, there are canonical valuations of root datum on $\mathcal{G} = G(k)$. The theory of étale descent ([BT2, §5]) reduces the question to the case of a quasi-split G. In that case, a valuation of root datum can be written down explicitly using a Chevalley-Steinberg system. This is done in [BT2, §4.1, §4.2]. One can also use [PR] and [L] for alternative treatments. Once this is taken care of, we have all the structures introduced in chapter 1.

But the bulk of chapter 2 is to show that now there are even more structures of algebro-geometric nature: the abstract bounded subgroups introduced in chapter 1 underly canonical group schemes. The main result of chapter 2 is Theorem 0.1 (with $f(0) = 0$). It is this part that we are able to replace and generalize.

We remark that the results of Bruhat-Tits are more general in that they allows fields with non-discrete valuations. Also, their theory of schematic root datum works over fairly general integral domains (hence gives a nice construction of Chevalley schemes over $\mathbb{Z}$, for example). Our goal is only to give a proof of Theorem 0.1 over a henselian discrete valuation ring.

0.4 The third purpose of this paper is to offer a replacement of the Moy-Prasad filtration for a torus. The original Moy-Prasad filtration has some anomalies (see 4.6) which are undesirable from the viewpoint of algebraic geometry. Also, the so-called Moy-Prasad isomorphism is valid only under certain tameness condition (which is satisfied in [MP1]). This isomorphism is needed for the proofs of [MP2]. Recently, DeBacker [D] has proved most results in [MP2] without using the isomorphism. However, the isomorphism is useful for many other purposes.

The minimal congruent filtration defined in this paper resolves all these problems. However, to make our theory more applicable, we axiomize a class of filtrations, called the schematic admissible filtrations, which include both the original Moy-Prasad filtration and the minimal congruent filtration. We prove the main theorem for all schematic admissible filtrations.

0.5 This paper is organized as follows. Section 1 summarizes some notations. In section 2, we collect facts about smooth models and smoothening. Section 3 contains the key lemmas in algebraic geometry, formulated in an axiomatic setting. These lemmas are applied in sections 7 and 8 to derive the main theorem, using smooth models of root subgroups and torus constructed in section 4-6. Section 9 contains various additional results, including some remark on Schneider-Stuhler theory.

In section 10, we examine the compact groups used in the author's earlier work on construction of tame supercuspidal representations [Y]. It turns out that these groups are not necessarily of the form $G(k)_{x,f}$, but they also admits canonical smooth models. Moreover, the inducing representations on these compact groups have a very simple and special form, indicating that they might be studied by algebraic geometry.

0.6 **Acknowledgment** The author would like to thank G. Prasad for his continual interests in this work. He is also grateful to A.-M. Aubert and C. Cunningham, as their work motivated him to write up the current article. Discussions with J. Adler, Aubert, C.-L. Chai, Cunningham, S. DeBacker, D. Prasad and G. Prasad are gratefully acknowledged. The work was first presented in a workshop of the Pacific Institute for the Mathematical Sciences, Canada, and written during the author's visit to the Institute for Mathematical Sciences, Singapore. The author thanks these institutes and the organizers for their supports and hospitality.

§1 Notations

1.1 Let k be a field with a henselian discrete valuation $\mathrm{ord} : k^\times \to \mathbb{Z}$. Let $\mathcal{O}$ be the ring of integers in k. We assume that the residue field κ of $\mathcal{O}$ is perfect. We denote by $\tilde{\mathcal{O}}$ the strict henselization of $\mathcal{O}$, and $\tilde{k}$ the field of fractions of $\tilde{\mathcal{O}}$. Finally, π is a fixed prime element of $\mathcal{O}$. The valuation is normalized so that $\mathrm{ord}(\pi) = 1$.

1.2 For a scheme $\underline{X}$ over $\mathrm{Spec}\,\mathcal{O}$, we denote by $\underline{X}_k$ the generic fiber $\underline{X} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,k$ of $\underline{X}$, and $\underline{X}_\kappa$ the special fiber $\underline{X} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\kappa$. Similarly, for an $\mathcal{O}$-algebra R, we set $R_k = R \otimes_{\mathcal{O}} k$, $R_\kappa = R \otimes_{\mathcal{O}} \kappa$.

We remind the reader that an algebraic variety over a field k is a separated and reduced scheme of finite type over k. In particular, the term algebraic group over a field k means a smooth group scheme over k. The neutral component of an algebraic group X over k is denoted by X°.

When we deal with varieties over κ, we sometimes abuse the language by identifying a variety X over κ with $X(\kappa)$. We only do this when κ is algebraically closed. It should cause no confusion.

We recall that a group scheme $\underline{X}$ over $\mathcal{O}$ is called *connected* if both $\underline{X}_k$ and $\underline{X}_\kappa$ are connected. A disconnected group scheme may have a connected underlying scheme. The neutral component $\underline{X}^\circ$ of a smooth group scheme $\underline{X}/\mathcal{O}$ is the open subscheme of $\underline{X}$ whose underlying set is $(\underline{X}_k)^\circ \cup (\underline{X}_\kappa)^\circ$.

1.3 If $\underline{G}$ is a group scheme over $\mathcal{O}$, its n^{th} *congruence subgroup* ($n \geq 1$) is defined to be

$$\Gamma(\pi^n, \underline{G}) := \ker\big(\underline{G}(\mathcal{O}) \to \underline{G}(\mathcal{O}/\pi^n\mathcal{O})\big).$$

We also put $\Gamma(\pi^0, \underline{G}) = \underline{G}(\mathcal{O})$.

1.4 Let G/k be a connected reductive group. Let S be a maximal k-split torus of G, T the centralizer of S in G (when G is quasi-split, T is a maximal torus of G), $\Phi = \Phi(G, S)$ the relative root system. For each $a \in \Phi$, let U_a be the corresponding root subgroup. We put $U_0 = T$.

Assume that G is quasi-split. Then there are valuations on the root datum $(T(k), \{U_a(k)\}_{a\in\Phi})$ by [BT1, 4.1, 4.2] or [PR] or [L]. Hence we have $\mathcal{B} = \mathcal{B}(G/k)$ the associated (reduced) Bruhat-Tits building. Moreover, any point x on $A(S)$ (the apartment on $\mathcal{B}$ attached to S) determines a valuation on $(T(k), \{U_a(k)\}_{a\in\Phi})$. This means that for each $a \in \Phi$, the group $U_a(k)$ is filtered by subgroups $U_a(k)_{x,r}$, $r \in \mathbb{R}$.

Many applications require us to apply [BT1] not to the root datum $(T(k), \{U_a(k)\}_{a\in\Phi})$, but to a variant $(\mathcal{T}, \{U_a(k)\}_{a\in\Phi})$, where $\mathcal{T}$ is a suitable subgroup of $T(k)$. There may be many possible choices of $\mathcal{T}$ (see [BT1, 6.1.2 (12), 6.2.11]). But the following two choices are most useful.

1.4.1 $\mathcal{T} = T(k) \cap G(k)^1$, where $G(k)^1 = \{g \in G(k) : \operatorname{ord}\chi(g) = 0 \text{ for all } \chi \in \operatorname{Hom}_k(G, \mathbb{G}_m)\}$ (see [BT2, 4.2.16]). This is most useful to study maximal bounded subgroups of $G(k)$.

1.4.2 $\mathcal{T} = T(k) \cap \mathcal{G}$, where $\mathcal{G}$ is the subgroup generated by $\underline{T}^{\mathrm{NR}}(\mathcal{O})$, and $U_a(k)$ for all $a \in \Phi$. Here, $\underline{T}^{\mathrm{NR}}$ is the Néron-Raynaud model of T. This is most useful for studying the (connected) parahoric subgroups.

1.4.3 In this paper, we work primarily with the choice of 1.4.2. The choice of 1.4.1 is relevant in the following two ways.

(i) The bounded subgroups U_f, P_f, U_Ω, P_Ω, $\widehat{P}_\Omega$ constructed in [BT1, 6.4, 7.1] can be formed with both choices. The group formed using 1.4.2 is contained in the same group formed using 1.4.1, and is of finite index in the latter.

(ii) The group $\widehat{P}_\Omega$ (see [BT1, 7.1]) made using 1.4.1 is precisely the subgroup of $G(k)$ fixing Ω pointwise, when Ω is regarded as a subset of the *enlarged* building.

1.5 Let $\tilde{\mathbb{R}}$ be the totally ordered monoid $\mathbb{R} \cup \{r+ : r \in \mathbb{R}\} \cup \{\infty\}$ as defined in [BT1, 6.4.1]. Whenever we have a decreasing filtration $\{F_r\}$ indexed by $r \in \mathbb{R}$ (resp. $r \in \mathbb{R}_{\geq 0}$), we extend it to a filtration indexed by $r \in \tilde{\mathbb{R}}$ (resp $r \in \tilde{\mathbb{R}}_{r \geq 0}$) as usual by putting $F_{r+} = \bigcup_{s>r} F_s$, $F_\infty = \bigcap_r F_r$. We say that r is a *jump* for the filtration $\{F_r\}$ if $F_r \neq F_{r+}$.

1.6 If Φ is a root system, Φ^{nd} will denote the (reduced) root system consisting of non-divisible roots in Φ, and $\widehat{\Phi} = \Phi \cup \{0\}$. If Φ^+ is a system of positive roots in Φ, Φ^- is simply $-\Phi^+$.

§2 Modifying integral models

Through out this section, X is an *affine* scheme of finite type over k.

2.1 **Definition** By an *integral model* (or simply *model*) of X, we mean an $\mathcal{O}$-scheme $\underline{X}$ of the form $\operatorname{Spec} A$ such that $A \subset \Gamma(X, \mathcal{O}_X)$ is a sub-$\mathcal{O}$-algebra of finite type over $\mathcal{O}$ with $A[\pi^{-1}] = \Gamma(X, \mathcal{O}_X)$. A model $\underline{X}$ is *smooth* if $\underline{X} \to \operatorname{Spec} \mathcal{O}$ is smooth.

Thus a model $\underline{X}$ of X is *flat* over $\operatorname{Spec} \mathcal{O}$, and has generic fiber X. Notice that for simplicity, we have restricted our attention to *affine* schemes in the above definition. This section reviews various properties about models and ways of modifying a model into new ones.

2.2 **Proposition (Uniqueness, [BT2, 1.7])** *Assume that X is smooth over k. Let $\underline{X}_1$ and $\underline{X}_2$ be smooth models of X such that $\underline{X}_1(\tilde{\mathcal{O}}) = X_2(\tilde{\mathcal{O}})$. Then $\underline{X}_1 = \underline{X}_2$. In fact, putting $W = \underline{X}_1(\tilde{\mathcal{O}}) = \underline{X}_2(\tilde{\mathcal{O}})$, we have*

$$\Gamma(\underline{X}_1, \mathcal{O}_{\underline{X}_1}) = \Gamma(\underline{X}_2, \mathcal{O}_{\underline{X}_2}) = \{f \in \Gamma(X, \mathcal{O}_X) \mid f(W) \subset \tilde{\mathcal{O}}\}.$$

2.3 **Proposition (Extension principle, [BT2, 1.7])** *Assume that $\underline{X}$ and $\underline{Y}$ are models of X and Y, $\underline{X}$ is smooth, and $\phi : X \to Y$ is a k-morphism such that $\phi(X(\tilde{\mathcal{O}})) \subset Y(\tilde{\mathcal{O}})$, then ϕ extends to a unique $\mathcal{O}$-morphism $\underline{X} \to \underline{Y}$.*

2.4 **Proposition (Étale descent, [L, 10.9])** *Suppose that k' is an unramified Galois extension of k, with ring of integers $\mathcal{O}'$. Let $\underline{X}'/\mathcal{O}'$ be a model of $X \otimes_k k'$ such that $\Gamma(\underline{X}', \mathcal{O}_{\underline{X}'})$ is a $\mathrm{Gal}(k'/k)$-stable sub-algebra of $\Gamma(X \otimes_k k', \mathcal{O}_{X\otimes_k k'})$. Then $\underline{X}'/\mathcal{O}'$ descends to a model $\underline{X}/\mathcal{O}$ canonically. If $\underline{X}'/\mathcal{O}'$ is smooth, so is $\underline{X}/\mathcal{O}$.*

2.5 **Restriction of scalars** Let $\mathcal{O}'$ be the ring of integers in a finite extension k' of k. If $\underline{X}'/\mathcal{O}'$ is a model of X'/k', then the Weil restriction $\underline{X} = \mathrm{Res}_{\mathcal{O}'/\mathcal{O}}\, \underline{X}'$ is a model of $X = \mathrm{Res}_{k'/k}\, X'$.

Lemma (i) *If $\underline{X}'/\mathcal{O}'$ is smooth, so is $\underline{X}/\mathcal{O}$.*

(ii) *If $\underline{X}'/\mathcal{O}'$ is a connected group scheme, so is $\underline{X}/\mathcal{O}$.*

(i) is [E, 2.2], and (ii) follows from [ELL, Proof of Theorem 1].

2.6 **Schematic closure** Let $\underline{X}$ be a model of X and Z be a closed sub-scheme of X. The schematic closure $\underline{Z}$ of Z in $\underline{X}$ is the smallest closed sub-scheme of $\underline{X}$ containing Z.

Lemma *The schematic closure $\underline{Z}$ is a model of Z. It is the unique model that represents the following functor on the category of flat $\mathcal{O}$-algebras:*

$$R \mapsto Z(R_k) \cap \underline{X}(R),$$

where the intersection is taken in $X(R_k)$.

PROOF. Let A_k (resp. B_k, resp. A) be the affine ring of X (resp. Z, resp $\underline{X}$). Then $\underline{Z}$ is affine, and its affine ring B is the image of A in the composition $A \hookrightarrow A_k \twoheadrightarrow B_k$. It is clear that $\underline{Z}$ is flat of finite type over $\mathcal{O}$, and is a model of Z. The statement about the functor of points is easy to check. ∎

2.7 **Dilatation** Let $\underline{X}$ be a model of X and Y a closed sub-scheme of $\underline{X}_\kappa$. The functor on the category of flat $\mathcal{O}$-algebra

$$R \mapsto \left\{x \in \underline{X}(R) \;\middle|\; x_\kappa : \mathrm{Spec}(R_\kappa) \to \underline{X}_\kappa \text{ factors through } Y \hookrightarrow \underline{X}_\kappa\right\}$$

is representable by a model $\underline{X}'$ of X, called the *dilatation* of Y on $\underline{X}$.

See [BLR, §3.2] for the construction of $\underline{X}'$. We remark that [BLR] treats more general (not necessarily affine) schemes over $\mathcal{O}$ but it is clear from the construction that the dilatation on an affine scheme is again affine. We also refer to [BLR, §3.2, Props. 2 and 3] for the following fact:

Lemma *Let $\underline{X}'$ be the dilatation of Y on $\underline{X}$.*

(i) *If $\underline{X}/\mathcal{O}$ and Y/κ are smooth, so is $\underline{X}'/\mathcal{O}$.*

(ii) *If $\underline{X}/\mathcal{O}$ is a group scheme and Y/κ is a closed subgroup of $\underline{X}_\kappa$, then $\underline{X}'/\mathcal{O}$ is naturally a group scheme.*

(iii) *Let $\underline{U}$ be an open sub-scheme of $\underline{X}$, and $V = \underline{U} \cap Y$, then the dilatation $\underline{U}'$ of V on $\underline{U}$ is an open sub-scheme of $\underline{X}'$.*

2.8 **Congruence subgroups** Let $\underline{X}$ be a smooth group scheme over $\mathcal{O}$ with generic fiber X. Then for each $n \geq 0$, there exists a smooth model $\underline{X}^{(n)}$ such that $\underline{X}^{(n)}(\tilde{\mathcal{O}}) = \Gamma(\pi^n, \underline{X} \otimes \tilde{\mathcal{O}})$. Moreover,

(i) $\big(\underline{X}^{(n)}\big)^{(m)} = \underline{X}^{(n+m)}$ for all $n, m \geq 0$.

(ii) $\operatorname{Lie} \underline{X}^{(n)} = \pi^n \operatorname{Lie} \underline{X}$ for all $n \geq 0$.

We first construct $\underline{X}^{(1)}$ as the dilatation of the trivial subgroup of $\underline{X}_\kappa$ on $\underline{X}$. By Lemma 2.7, this is a smooth model with the desired set of integral points.

Write $\Gamma(\underline{X}, \mathcal{O}_{\underline{X}}) = \mathcal{O}[f_1, \ldots, f_n]$. We may and do assume that $f_i(e) = 0$ (e being the identity section) for all i and $\{f_i \bmod \pi\}_i$ generate the ideal defining the trivial subgroup of $\underline{X}_\kappa$. Then

$$\underline{X}^{(1)} = \operatorname{Spec} \mathcal{O}[f_1/\pi, \ldots, f_n/\pi]$$

by the construction of dilatation in [BLR, §3.2]. From this, it follows immediately that we have

$$\operatorname{Lie} \underline{X}^{(1)} = \pi \operatorname{Lie} \underline{X} \qquad \text{and} \qquad \Gamma(\pi^n, \underline{X}^{(1)}) = \Gamma(\pi^{n+1}, \underline{X})$$

for all $n \geq 0$.

Now we can construct $\underline{X}^{(n)}$ inductively as follows: put $\underline{X}^{(0)} = \underline{X}$, and $\underline{X}^{(n+1)}$ to be the dilatation of the trivial subgroup of $\underline{X}^{(n)}_\kappa$ on $\underline{X}^{(n)}$ for $n \geq 0$. It is easy to verify that $X^{(n)}$ satisfies all the claimed properties.

Lemma *For $n \geq 1$, $(\underline{X}^{(n)})_\kappa$ is a vector group.*

Proof. Let $L = \operatorname{Lie} \underline{X}$. There is a functorial isomorphism

$$\Gamma(\pi^a, \underline{X}) / \Gamma(\pi^b, \underline{X}) \simeq (\pi^a L)/(\pi^b L)$$

whenever $2a \geq b \geq a \geq 1$. This shows that $\underline{X}^{(n)}(\kappa) \simeq \Gamma(\pi^n, \underline{X}) / \Gamma(\pi^{n+1}, \underline{X}) \simeq (\pi^n L)/(\pi^{n+1} L)$. ■

2.9 **Group smoothening** Let $\underline{X}$ be an affine group scheme such that $X = X_k$ is a smooth group scheme over k. Then there exists a unique model $\underline{X}^{\mathrm{sm}}$, called *the smoothening of* $\underline{X}$, such that $\underline{X}^{\mathrm{sm}}$ is smooth and $\underline{X}^{\mathrm{sm}}(\tilde{\mathcal{O}}) = \underline{X}(\tilde{\mathcal{O}})$.

The existence follows from [BLR, 7.1]. There, the smoothening is carried out for $\underline{X}$ not necessarily affine. However, in our case, $\underline{X}^{\mathrm{sm}}$ is indeed affine because the smoothening process in [BLR] is a series of dilatations, which take affine schemes to affine schemes. The uniqueness then follows from Prop. 2.2.

***2.10* Intersection** Let $\underline{X}_1$ and $\underline{X}_2$ be models of X. Then there is a model $\underline{X}$ of X such that $\underline{X}(\tilde{\mathcal{O}}) = \underline{X}_1(\tilde{\mathcal{O}}) \cap \underline{X}_2(\tilde{\mathcal{O}})$.

We embed X into $X \times X$ diagonally, and take the schematic closure of X in $\underline{X}_1 \times_{\mathcal{O}} \underline{X}_2$ to obtain $\underline{X}$.

***2.11* Fixed points** Let F be a finite group acting on a model $\underline{X}$ of X by automorphisms over $\mathcal{O}$. Then $\underline{X}^F$ is affine with generic fiber X^F, and it represents the functor $R \mapsto \underline{X}(R)^F$ on the category of $\mathcal{O}$-algebras.

Proposition (Edixhoven, [E, 3.4]) *If $\underline{X}/\mathcal{O}$ is smooth and $\#F$ is invertible on $\mathcal{O}$, then $\underline{X}^F/\mathcal{O}$ is a smooth model of X^F.*

§3 Lemmas

***3.1* A criterion for open immersion** The following result is very useful in this paper.

Lemma *Let $\phi : \underline{X} \to \underline{Y}$ be a morphism of affine smooth integral $\mathcal{O}$-schemes. Assume that $\underline{X}_\kappa$ is connected, and*

- (i) *ϕ_k is an open immersion onto an open sub-variety of the form $(\underline{Y}_k)_h$, $h \in k[\underline{Y}_k]$;*
- (ii) *ϕ_κ is dominant.*

Then ϕ is an open immersion onto an open sub-scheme of the form $\underline{Y}_{h'}$, $h' \in \mathcal{O}[\underline{Y}]$.

PROOF. Let A (resp. B) be the affine ring of $\underline{X}$ (resp. $\underline{Y}$). We may and do assume that $h \in B$ and $\pi^{-1}h \notin B$. Let $\bar{h} \in B/\pi B$ be the reduction of h modulo π. The equation $\bar{h} = 0$ defines a closed subset W of $\underline{Y}_\kappa$. Let $Z = \phi_\kappa^{-1}(W) \subset \underline{X}_\kappa$. We notice that $W \neq \underline{Y}_\kappa$ since $\bar{h} \neq 0$ and $\underline{Y}_\kappa$ is smooth (hence reduced), and $Z \neq \underline{X}_\kappa$ since ϕ_κ is dominant. Consequently, Z is of codimension ≥ 2 in $\underline{X}$.

Let $\underline{X}'$ be the open sub-scheme $\underline{X} \setminus Z$ and let ϕ' be the restriction $\underline{X}' \to \underline{Y}_h$. We observe that $\Gamma(\underline{X}', \mathcal{O}_{\underline{X}'}) = A$, because a rational function on $\underline{X}$ regular away from the closed subset Z, which is of codimension ≥ 2, is regular on $\underline{X}$. The canonical bijection $\mathrm{Hom}(\underline{X}', \operatorname{Spec} B_h) = \mathrm{Hom}(B_h, \Gamma(\underline{X}', \mathcal{O}_{\underline{X}'}))$ gives us a ring morphism $B_h \to A$, which shows that ϕ' actually extends to a morphism $\underline{X} \to \underline{Y}_h$. Of course, this morphism is simply ϕ. Therefore, Z is empty.

Now the following claim applies to $B_h \to A$ (as a morphism between $\mathcal{O}$-modules) and shows that $B_h \simeq A$ and $\underline{X} \simeq \underline{Y}_h$. The lemma is proved. ∎

Claim. *Let $\phi : M \to N$ be a morphism of flat $\mathcal{O}$-modules such that*

- (i) *$\phi \otimes k$ is bijective,*
- (ii) *$\phi \otimes \kappa$ is injective.*

Then ϕ is an isomorphism.

PROOF. Since M is flat, $M \hookrightarrow M \otimes k \simeq N \otimes k$ and thus ϕ is injective. Consider the exact sequence

$$0 \to M \to N \to M/N \to 0.$$

By (i), M/N is a torsion module. By (ii) and the flatness of N, $\mathrm{Tor}^1(M/N, \kappa) = 0$. But $\mathrm{Tor}^1(M/N, \kappa)$ is just the submodule of M/N of π-torsion elements. Therefore, we have $M/N = 0$. ∎

3.2 **Setting** For the rest of this section up to 3.5, k is assumed to be strictly henselian for simplicity. Let G be a connected affine algebraic group over k, $\{U_i\}_{i=1}^s$ a collection of algebraic subgroups of G defined over k. We assume that the multiplication morphism

$$\prod_{i=1}^{s} U_i \to G, \qquad (u_1, \ldots, u_s) \mapsto u_1 \cdots u_s,$$

is an open immersion onto an open sub-variety of the form G_h, where $h \in k[G]$ is a regular function on G.

Let $\mathcal{K} = \{K_i\}_{i=1}^s$ be a collection of groups such that $K_i \subset U_i(k)$ for each i. We say that $\mathcal{K}$ is *schematic* if for each i, there is a (unique) smooth model $\underline{U}_{i,\mathcal{K}}$ of U_i such that $\underline{U}_{i,\mathcal{K}}(\mathcal{O}) = K_i$.

For an schematic $\mathcal{K} = \{K_i\}$, let $G(k)_{\mathcal{K}}$ be the subgroup of $G(k)$ generated by $\{K_i\}_{i=1}^s$. We will consider the following properties about $\mathcal{K}$:

- **P1** There is a smooth model $\underline{G}_{\mathcal{K}}$ of G such that $\underline{G}_{\mathcal{K}}(\mathcal{O}) = G(k)_{\mathcal{K}}$.
- **P2** Property **P1** holds, and for each i, the natural morphism $\underline{U}_{i,\mathcal{K}} \to \underline{G}_{\mathcal{K}}$ is a closed immersion.
- **P3** Property **P1** holds, and the multiplication morphism $\prod_{i=1}^s \underline{U}_{i,\mathcal{K}} \to \underline{G}_{\mathcal{K}}$ is an open immersion.
- **P3′** Property **P1** holds and the multiplication morphism $\prod_{i=1}^s \underline{U}_{i,\mathcal{K}} \to \underline{G}_{\mathcal{K}}$ is an open immersion which induces an isomorphism on the special fibers.
- **P3″** Property **P1** holds, and the multiplication map $\prod_{i=1}^s K_i \to G(k)_{\mathcal{K}}$ is surjective modulo $\Gamma(\pi, \underline{G}_{\mathcal{K}})$.

3.3 **Lemma** *We have the follow implications.*

(i) **P3′** *implies* **P3** *and* **P3″**.
(ii) **P3″** *imply* **P3′**, *provided that all* $\underline{U}_{i,\mathcal{K}}$ *are connected.*
(iii) **P3** *implies* **P2**.

PROOF. (i) is obvious. (ii) is clear from Lemma 3.1.

To prove (iii), let $\underline{U}_i'$ be the schematic closure of U_i in $\underline{G}_{\mathcal{K}}$. Then **P3** implies that $\underline{U}_{i,\mathcal{K}} \to \underline{U}_i'$ is an open immersion. This implies that the neutral component of $(\underline{U}_i')_\kappa$ is smooth over κ, hence $(\underline{U}_i')_\kappa$ is smooth over κ. It follows that $\underline{U}_i'$ is smooth over $\mathcal{O}$. (iii) now results from 2.2, as $\underline{U}_i'(\mathcal{O}) = K_i = \underline{U}_{i,\mathcal{K}}(\mathcal{O})$. ∎

3.4 We now consider two schematic collections $\mathcal{K}' = \{K_i'\}$ and $\mathcal{K}'' = \{K_i''\}$. We assume that $K_i'' \subset K_i'$ for all i.

Assume $\mathcal{K} = \mathcal{K}'$ satisfies **P1**. By Prop. 2.3, there are natural morphisms $\underline{U}_{i,\mathcal{K}''} \to \underline{G}_{\mathcal{K}'}$ for all i. Let $V_i(\kappa)$ be the image of $\underline{U}_{i,\mathcal{K}''}(\kappa)$ in $\underline{G}_{\mathcal{K}'}(\kappa)$, and $H(\kappa)$ the subgroup of $\underline{G}_{\mathcal{K}'}(\kappa)$ generated by $V_i(\kappa)$, for all i.

3.4.1 **Lemma** *Assume*

(i) $\mathcal{K} = \mathcal{K}'$ *satisfies* **P1.**

(ii) $G(k)_{\mathcal{K}''} \supset \Gamma(\pi, \underline{G}_{\mathcal{K}'})$.

(iii) $H(\kappa)$ *is Zariski closed.*

Then $\mathcal{K} = \mathcal{K}''$ *also satisfies* **P1.**

We remark that assumption (iii) of the preceding lemma is satisfied when $V_i(\kappa)$ is connected for all i. This follows from [Sp, Prop. 2.2.6]

PROOF. By (iii), $H(\kappa)$ underlies an algebraic subgroup H of $(\underline{G}_{\mathcal{K}'})_\kappa$. By (ii), $G(k)_{\mathcal{K}''}$ is the inverse image of $H(\kappa)$ in $\underline{G}_{\mathcal{K}'}(\mathcal{O}) \to \underline{G}_{\mathcal{K}'}(\kappa)$. Therefore, taking $\underline{G}_{\mathcal{K}''}$ to be the dilatation of H on $\underline{G}_{\mathcal{K}'}$, we see that **P1** is satisfied for $\mathcal{K} = \mathcal{K}''$. ∎

3.4.2 **Lemma** *Maintain the assumptions of Lemma 3.4.1. Moreover, assume*

(iv) **P3** *holds for* $\mathcal{K} = \mathcal{K}'$. *Denote the image of the open immersion* $\prod_{i=1}^s \underline{U}_{i,\mathcal{K}'} \to \underline{G}_{\mathcal{K}'}$ *by* $\underline{U}$.

(v) $H(\kappa) \cap \underline{U}(\kappa)$ *is the image of* $\prod_{i=1}^s \underline{U}_{i,\mathcal{K}''}(\kappa) \to H(\kappa)$.

Then **P3** *holds for* $\mathcal{K} = \mathcal{K}''$.

PROOF. By Lemma 3.4.1, $\underline{G}_{\mathcal{K}''}$ is the dilatation of H on $\underline{G}_{\mathcal{K}'}$. By Lemma 2.7 (iii), the dilatation $\underline{U}'$ of $H(\kappa) \cap \underline{U}(\kappa)$ on $\underline{U}$ is an open sub-scheme of $\underline{G}_{\mathcal{K}''}$.

By [BLR, §3.2, Prop. 2 (d)], $\underline{U}'$ is isomorphic to the direct product of the dilatation of $V_i(\kappa)$ on $\underline{U}_{i,\mathcal{K}'}$, which is nothing but $\underline{U}_{i,\mathcal{K}''}$ by Lemma 2.7 (i) and Prop 2.2. The isomorphism is just the product morphism, since it is the product morphism on the generic fiber. The lemma is proved. ∎

3.5 To apply Lemma 3.4.1, we need to know $\Gamma(\pi, \underline{G}_{\mathcal{K}'})$. This can often be done by using the following lemma.

Lemma *Assume that* **P1** *and* **P3** *hold for* $\mathcal{K} = \mathcal{K}'$. *Then the multiplication map*

$$\prod_{i=1}^{s} \Gamma(\pi, \underline{U}_{i,\mathcal{K}'}) \to \Gamma(\pi, \underline{G}_{\mathcal{K}'})$$

is a bijection. Therefore, $G(k)_{\mathcal{K}''} \supset \Gamma(\pi, \underline{G}_{\mathcal{K}'})$ *if* $K''_i \supset \Gamma(\pi, \underline{U}_{i,\mathcal{K}'})$ *for all* i.

PROOF. It is clear that we have an injection. We need to show that this injection is surjective.

Let $m : \prod_{i=1}^{s} \underline{U}_{i,\mathcal{K}'} \to \underline{G}_{\mathcal{K}'}$ be the open immersion asserted by **P3**, $\underline{U}$ its image, and $\underline{Z}$ the complement of $\underline{U}$ in $\underline{G}_{\mathcal{K}'}$, as a closed sub-scheme with reduced induced structure.

Now let $g : \operatorname{Spec} \mathcal{O} \to \underline{G}_{\mathcal{K}'}$ be an $\mathcal{O}$-valued point of $\underline{G}_{\mathcal{K}'}$ such that $g \in \Gamma(\pi, \underline{G}_{\mathcal{K}'})$. Suppose that $g(\operatorname{Spec} k) \in \underline{G}_{\mathcal{K}'}(k) = G(k)$ lies in $\underline{Z}(k)$. Then $g(\operatorname{Spec} \mathcal{O})$, which is the closure of $g(\operatorname{Spec} k)$, would lie in $\underline{Z}$. However, $g(\operatorname{Spec} \kappa)$ is the identity on $\underline{G}_{\mathcal{K}'}(\kappa)$ and does not belong to $\underline{Z}(\kappa)$. Therefore, $g(\operatorname{Spec} k)$ lies in $\underline{U}$. Since $g(\operatorname{Spec} \kappa)$ also lies in $\underline{U}$, g factors through the open sub-scheme $\underline{U}$, in other words, $g \in \underline{U}(\mathcal{O})$.

It follows that g factorizes uniquely as $\prod_{i=1}^{s} u_i$, $u_i \in \underline{U}_{i,\mathcal{K}'}(\mathcal{O})$. In order that $g \in \Gamma(\pi, \underline{G}_{\mathcal{K}'})$, it is necessary and sufficient that $u_i \in \Gamma(\pi, \underline{U}_{i,\mathcal{K}'})$ for each i. The lemma is proved. ∎

3.6 The above setting will be applied to a quasi-split connected reductive group G over k, with the help of the following lemma.

Lemma *For any system* $\Phi^{\mathrm{nd}+}$ *of positive roots in* Φ^{nd}, *the multiplication morphism*

$$\Big(\prod_{a \in \Phi^{\mathrm{nd}-}} U_a\Big) \times U_0 \times \Big(\prod_{a \in \Phi^{\mathrm{nd}+}} U_a\Big) \to G$$

is an open immersion onto an open sub-scheme of the form G_h, $h \in k[G]$, *where the products* $\prod_{a \in \Phi^{\mathrm{nd}\pm}}$ *can be taken in any order.*

This follows from the proof of [St, §5, Theorem 7]. Notice that the proof of [St] is done in the context of a split semi-simple algebraic group over k, but it also works for a quasi-split reductive group over k. The lemma can also be deduced easily from [BT2, Théorème 2.2.3].

§4 Admissible filtrations on tori

In this section, let T/k be a torus. We will denote by $\underline{T}^{\mathrm{NR}}/\mathcal{O}$ the *Néron-Raynaud model* of T/k (also known as the *connected Néron model* of T/k, see [CY]).

4.1 **Induced tori** A torus R over k is called *induced* if $X_*(R)$ has a $\mathbb{Z}$-basis which is permuted by $\mathrm{Gal}(\bar{k}/k)$. Equivalently, R is isomorphic to $\prod_{i=1}^m \mathrm{Res}_{k_i/k}\,\mathbb{G}_{\mathrm{m}}$ for a suitable set of finite separable extensions $k_1/k, \dots, k_m/k$.

Every torus can be embedded into an induced torus.

4.2 **The Moy-Prasad filtration for a torus** The Moy-Prasad filtration on $T(k)$, also defined by Schneider-Stuhler, is a sequence $\{T(k)_r^{\mathrm{MP}}\}_{r\geq 0}$ of subgroups of $T(k)$ defined as follows. If $r = 0$, we simply put $T(k)_r^{\mathrm{MP}} = \underline{T}^{\mathrm{NR}}(\mathcal{O})$. Now suppose $r > 0$.

(i) If $T = R = \prod_{i=1}^m \mathrm{Res}_{k_i/k}\,\mathbb{G}_{\mathrm{m}}$ is an induced torus, we define $R(k)_r^{\mathrm{MP}}$ to be the following subgroup of $R(k) = \prod k_i^\times$:

$$R(k)_r^{\mathrm{MP}} := \Bigl\{(x_i) \in \prod k_i^\times \Bigm| \mathrm{ord}_k(x_i - 1) \geq r \text{ for each } i\Bigr\}.$$

Here, ord_k is the valuation on k_i extending ord on k.

(ii) In general, choose an induced torus R containing T, then we put

$$T(k)_r^{\mathrm{MP}} := R(k)_r^{\mathrm{MP}} \cap \underline{T}^{\mathrm{NR}}(\mathcal{O}).$$

It is easy to show that this is independent of the choice of R.

Our definition is a bit different from that in [MP2] or [SS], but it is easy to check that it is equivalent to theirs. There is also a Moy-Prasad filtration $\{\mathfrak{t}_r^{\mathrm{MP}}\}_{r\in\mathbb{R}}$ on $\mathfrak{t} = \mathrm{Lie}\,T$. We recall the definition quickly: for $T = R$ as in (i) above, we set $\mathfrak{r}_r^{\mathrm{MP}} = \{(X_i) \in \prod k_i \mid \mathrm{ord}_k(X_i) \geq r \text{ for each } i\}$, where $\mathfrak{r} = \mathrm{Lie}\,R = \prod k_i$. In general, we embed T into an induced torus R as (ii) above, and set $\mathfrak{t}_r^{\mathrm{MP}} = \mathfrak{r}_r^{\mathrm{MP}} \cap \mathfrak{t}$.

4.3 **Admissible filtrations** Consider a functor (in the sense of **F3** below) $T/k \mapsto \{T(k)_r^\dagger\}_{r\geq 0}$ from tori over henselian local fields to sequence of groups. We say that this functor is an *admissible filtration* if the following conditions are satisfied.

F0 $T(k)_s^\dagger \subset T(k)_r^\dagger \subset T(k)$ for all $s \geq r \geq 0$.

F1 $T(k)_r^\dagger \subset T(k)_r^{\mathrm{MP}}$ for each $r \geq 0$, and $T(k)_0^\dagger = T(k)_0^{\mathrm{MP}}$, $T(k)_{0+}^\dagger = T(k)_{0+}^{\mathrm{MP}}$.

F2 $T(k)_r^\dagger = T(k)_r^{\mathrm{MP}}$ if T is induced.

F3 If $T \to T'$ is a morphism between tori over k, then $T(k) \to T'(k)$ maps $T(k)_r^\dagger$ into $T'(k)_r^\dagger$ for all $r \geq 0$.

F4 If k'/k is unramified, then $T(k)_r^\dagger = T(k')_r^\dagger \cap T(k)$.

These conditions are designed so that an admissible filtration is good in the sense of 6.1.

4.4 **Additional conditions** Let $T/k \mapsto \{T(k)_r^\dagger\}$ be admissible. We will say that it is *schematic* if the following three conditions are satisfied.

S1 For each $r \geq 0$, there is a unique smooth group scheme $\underline{T}_r^\dagger$ over $\mathcal{O}$ such that $\underline{T}_r^\dagger(\tilde{\mathcal{O}}) = T(\tilde{k})_r^\dagger$.

S2 $\Gamma(\pi, \underline{T}_r^\dagger) \subset T(k)_{r+1}^\dagger$ for all $r \geq 0$.

S3 For any T/k, there is a strictly increasing sequence $\{r_i\}_{i\geq 1}$ of non-negative real number such that $r_i \to \infty$ as $i \to \infty$, and $T(k)_r^\dagger \neq T(k)_{r+}^\dagger$ only when $r = r_i$ for some i.

A schematic filtration is called *connected* if the following condition is satisifed:

CN For each $r \geq 0$, the scheme $\underline{T}_r^\dagger$ is connected.

A schematic filtration is called *congruent* if the following condition is satisfied:

CG $\Gamma(\pi, \underline{T}_r^\dagger) = T(k)_{r+1}^\dagger$ for all $r \geq 0$.

4.4.1 **Lemma** *Assume conditions* **S1**, **CN**, *and* **CG**. *Then conditions* **S2** *and* **S3** *are also satisifed. In fact, the following stronger condition is satisifed:*

S3′ *There are finitely many real numbers $s_1, \ldots, s_d \in [0, 1)$ such that $T(k)_r^\dagger \neq T(k)_{r+}^\dagger$ if and only if $r = s_i + n$ for some i and some $n \in \mathbb{Z}_{\geq 0}$.*

PROOF. **S2** is clear. If $T(k)_r^\dagger \neq T(k)_{r+}^\dagger$ and $r \in [0, 1)$, then $\dim T(k)_{r+}^\dagger / \Gamma(\pi, \underline{T}_0^\dagger) < \dim T(k)_r^\dagger / \Gamma(\pi, \underline{T}_0^\dagger)$ by **CN** and **CG**. It follows that there are at most $\dim T$ jumps in $[0, 1)$ for the filtration $\{T(k)_r^\dagger\}$. It is also clear that for any $n \in \mathbb{Z}_{\geq 0}$, $T(k)_r^\dagger \neq T(k)_{r+}^\dagger$ if and only if $T(k)_{r+n}^\dagger \neq T(k)_{(r+n)+}^\dagger$. Now S3′ is clear. ∎

4.4.2 **Lemma** *Assume that $T/k \mapsto \{T(k)_r^\dagger\}$ is admisible and schematic. Define $T(k)_r^{\dagger\circ}$ to be $(\underline{T}_r^\dagger)^\circ(\mathcal{O})$. Then $T/k \mapsto \{T(k)_r^{\dagger\circ}\}$ is admissible, schematic, and connected.*

PROOF. Properties **F0**, **F1**, **F4**, **S1** and **S3** are obvious. **F2** is easy (see Prop. 4.7.3 below). **F3** holds because the morphism $\underline{T}_r \to \underline{T}'_r$ must send the neutral component $(\underline{T}_r^\dagger)^\circ$ to the neutral component $(\underline{T}')_r^\dagger)^\circ$. Similarly, **S2** follows from **S2** for $\{T(k)_r^\dagger\}$ and Lemma 2.8. ∎

4.5 **Proposition** *The Moy-Prasad filtration $T/k \mapsto \{T(k)_r^{\mathrm{MP}}\}_{r\geq 0}$ is admissible and schematic.*

PROOF. Admissibility and **S3** are clear. We now verify **S1** by constructing $\underline{T}_r^{\mathrm{MP}}$. When $r = 0$, $\underline{T}_r^{\mathrm{MP}}$ is just $\underline{T}^{\mathrm{NR}}$. Suppose that $r > 0$.

Assume $T = \mathbb{G}_{\mathrm{m}}/k$. Then it suffices to let $\underline{T}_r^{\mathrm{MP}}$ to be the congruence model $(\mathbb{G}_{\mathrm{m}}/\mathcal{O})^{(n)}$, where $n = \lceil r \rceil$. Concretely, the affine ring of T is $k[X, Y]$ with the relation $XY = 1$, and the model

$$\operatorname{Spec} \mathcal{O}[\pi^{-n}(X - 1), \pi^{-n}(Y - 1)]$$

of T has the required properties for $\underline{T}_r^{\mathrm{MP}}$. Now we can proceed following the steps defining the Moy-Prasad filtrations.

(i) Assume that $T = R = \prod_{i=1}^m \mathrm{Res}_{k_i/k}\,\mathbb{G}_\mathrm{m}$ is induced. Then it is easy to see that the model $\prod_{i=1}^m \mathrm{Res}_{\mathcal{O}_i/\mathcal{O}}\big(\underline{\mathbb{G}_\mathrm{m}}\big)^{\mathrm{MP}}_{e_i r}$ of R has the required property for $\underline{R}_r^{\mathrm{MP}}$, where $\mathcal{O}_i$ is the ring of integers in k_i, and e_i is the ramification index of k_i/k.

(ii) For general T, take an induced torus R containing T. Then the schematic closure $\underline{T}'_r$ of T in $\underline{R}_r$ is a model of T satisfying $\underline{T}'_r(\tilde{\mathcal{O}}) = T(\tilde{k}) \cap R(\tilde{k})_r$. We then apply 2.10 to $\underline{T}'_r$ and $\underline{T}^{\mathrm{NR}}$, followed by smoothening (see 2.9) to obtain $\underline{T}_r^{\mathrm{MP}}$.

Next, we verify **S2**. Take an induced torus R containing T. By Prop. 2.3, there is a morphism $\underline{T}_r^{\mathrm{MP}} \to \underline{R}_r^{\mathrm{MP}}$. This shows that $\Gamma(\pi, \underline{T}_r^{\mathrm{MP}}) \subset \Gamma(\pi, \underline{R}_r^{\mathrm{MP}})$. However, it is easy to show that $\Gamma(\pi, \underline{R}_r^{\mathrm{MP}}) = R(k)_{r+1}^{\mathrm{MP}}$ (see Prop. 4.7.2). Therefore, $\Gamma(\pi, \underline{T}_r^{\mathrm{MP}}) \subset R(k)_{r+1}^{\mathrm{MP}} \cap T(k)_r^{\mathrm{MP}} = T(k)_{r+1}^{\mathrm{MP}}$. ∎

4.6 It is easy to show that $T/k \mapsto \{T(k)_r^{\mathrm{MP}}\}$ is not congruent. See, for example, [GHY, Lemma B.1]. C.-L. Chai pointed out to the author that it is not connected either, and provided the following example.

4.6.1 **Example.** For any torus T/k, define another filtration $\{T(k)'_r\}$ on $T(k)$ as follows: take an embedding $T \hookrightarrow R$ into an induced torus R, and set $T(k)'_r = T(k) \cap R(k)_r$. Notice that the difference with the Moy-Prasad filtration is that we don't intersect with $\underline{T}^{\mathrm{NR}}(\mathcal{O})$. The argument for Prop. 4.5 shows that there is a unique smooth model $\underline{T}'_r$ of T such that $\underline{T}'_r(\tilde{\mathcal{O}}) = T(\tilde{k})'_r$. We will show that $\underline{T}'_r$ often fails to be connected.

Let K/k be a finite Galois extension of degree e, $\Gamma = \mathrm{Gal}(K/k)$. Let $T = \ker(N : \mathrm{Res}_{K/k}\,\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m})$ be the corresponding norm one torus. Let $n \geq 0$ be be an integer such that $\Gamma^n \neq \Gamma^{n+}$, where $\{\Gamma^r\}_{r\geq 0}$ is the upper number filtration. Then $\underline{T}'_{\psi(n)/e}$ is disconnected, where ψ is the Herbrand function [Se, IV.3]. This follows from [Se, V.6].

Of course, this does not yet imply that $\underline{T}_r^{\mathrm{MP}}$ may be disconnected. C.-L. Chai has also shown the author how to analyze the effect of taking intersection with $\underline{T}^{\mathrm{NR}}(\mathcal{O})$, i.e. to study the disconnection for $\underline{T}_r^{\mathrm{MP}}$ for the above torus T. The calculation is more involved and not reproduced here.

4.6.2 **Example.** Assume that k is of equal-characteristic 2. Let $K/k = k(x)/k$ be the extension defined by the Eisenstein equation

$$x^4 + (\pi^4 + \pi)x^2 + \pi^3 x + \pi = 0.$$

Let $T = \ker(N : \mathrm{Res}_{K/k}\,\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m})$ be the corresponding norm one torus. Then $\underline{T}_{7/4}^{\mathrm{MP}}$ has component group $\mathbb{Z}/2\mathbb{Z}$. This can be checked by a direct computation.

4.6.3 It follows from Lemma 4.4.2 that $T/k \mapsto T(k)_r^{\mathrm{MP}\circ}$ is a connected filtration. It is the largest connected filtration. This construction was also observed by G. Prasad.

4.7 **A tameness assumption** The construction of $\underline{T}_r^{\mathrm{MP}}$ involves taking schematic closure and performing the smoothening process. This makes it difficult to say anything about $\underline{T}_r^{\mathrm{MP}}$ beyond its existence. It is useful to consider the following condition on T. Under this condition, we will give another construction of $\underline{T}_r^{\mathrm{MP}}$, and derive further properties about admissible filtrations.

4.7.1 **Condition (T)** There exists a tamely ramified finite extension k'/k such that $T \otimes_k k'$ is induced.

We can assume in addition that k'/k is Galois. This does not change the condition.

4.7.2 Assume condition (**T**). For simplicity of exposition (and without loss of generality), assume $k = \tilde{k}$ is strictly henselian. Then we have a canonical embedding $T \hookrightarrow R$, where $R = \mathrm{Res}_{k'/k}(T \otimes_k k')$. Notice that R is induced and hence there is an explicit construction of $\underline{R}_r^{\mathrm{MP}}$. Moreover, $F = \mathrm{Gal}(k'/k)$ acts on R and $R^F = T$. The action of F leaves $R(k)_r^{\mathrm{MP}}$ stable for all $r \geq 0$. Therefore, F acts on $\underline{R}_r^{\mathrm{MP}}$ by Prop. 2.3.

Proposition *With the above assumptions,* $\underline{T}_r^{\mathrm{MP}} = (\underline{R}_r^{\mathrm{MP}})^F$ *for* $r > 0$.

PROOF. Fix $r > 0$. By Prop. 2.11, it suffices to show that $R(k)_r^{\mathrm{MP}} \cap T(k) \subset \underline{T}^{\mathrm{NR}}(\mathcal{O})$. In fact, we claim that $R(k)_{0+}^{\mathrm{MP}} \cap T(k) = T(k)_{0+}^{\mathrm{MP}}$. Recalling that $T(k)_{0+}^{\mathrm{MP}}$ is the pre-image of $(R_{\mathrm{u}}\underline{T}_\kappa^{\mathrm{NR}})(\kappa)$ in $T(k)_0^{\mathrm{MP}} \to \underline{T}^{\mathrm{NR}}(\kappa)$, where R_{u} is the unipotent radical, we see that it suffices to prove the following statement: *let* G *be a connected affine algebraic group over* κ *and* $F \subset \mathrm{Aut}_\kappa(G)$ *a finite group of order invertible in* κ, $H = (G^F)^\circ$, *then* $(R_{\mathrm{u}}G)^F = R_{\mathrm{u}}H$. This is immediate from the following lemma and [PY, Theorem 2.1] (here we are applying this statement to a commutative G, for which [PY, Theorem 2.1] is not needed). ■

Lemma *Let* U *be a connected unipotent algebraic group over* κ *and* $F \subset \mathrm{Aut}_\kappa(U)$ *of order invertible on* κ. *Then* U^F *is connected.*

PROOF. We can find a connected, F-stable, Zariski closed subgroup U' of U such that U/U' is a vector group of positive dimension. Then we have an exact sequence $0 \to (U')^F \to U^F \to (U/U')^F \to 0$. Since $(U/U')^F \simeq (U/U')_F$ is clearly connected, we are done by induction on $\dim U$. ■

4.7.3 **Proposition** *Assume that* T *satisfies condition* (**T**). *Then we have*

(i) $\underline{T}_r^{\mathrm{MP}}$ *is connected for all* $r \geq 0$, *and* $(\underline{T}_r^{\mathrm{MP}})_\kappa$ *is unipotent for all* $r > 0$.

(ii) *The congruence subgroup* $\Gamma(\pi^n, \underline{T}_r^{\mathrm{MP}})$ *is equal to* $T(k)_{n+r}^{\mathrm{MP}}$.

(iii) $\operatorname{Lie}\underline{T}_r^{\mathrm{MP}} = \mathfrak{t}_r^{\mathrm{MP}}$.

Proof. (i) As the statements are clear for $r = 0$, we assume $r > 0$. Then the statements are easily checked when T is induced. In general, Proposition 4.7.2 tell us that $(\underline{T}_r^{\mathrm{MP}})_\kappa$ is simply $(\underline{R}_r^{\mathrm{MP}})^F$, hence the result follows from Lemma 4.7.2.

(ii) and (iii): the statements are easy to check when T is induced. In general, we use Proposition 4.7.2 and the following easy fact: for any group scheme $\underline{G}/\mathcal{O}$, any finite group $F \subset \operatorname{Aut}_{\mathcal{O}}(\underline{G})$, we have $\Gamma(\pi^n, \underline{G})^F = \Gamma(\pi^n, \underline{G}^F)$, and $\operatorname{Lie}(\underline{G}^F) = (\operatorname{Lie}\underline{G})^F$. ∎

4.7.4 **Lemma** *Let $T/k \mapsto \{T(k)_r^\dagger\}$ be an admissible filtration. Then $T(k)_r^\dagger = T(k)_r^{\mathrm{MP}}$ for all T satisfying (***T***), and all $r \geq 0$.*

Proof. By **F1** and **F4**, we may and do assume that $r > 0$ and k is strictly henselian. Let k'/k be a finite Galois tamely ramified extension such that $T \otimes_k k'$ is induced. Let R be the induced torus $\operatorname{Res}_{k'/k}(T \otimes_k k')$. There is a "norm" map $R \to T$. We claim that this maps $R(k)_r^{\mathrm{MP}}$ *onto* $T(k)_r^{\mathrm{MP}}$. Clearly, this claim and **F3** imply the lemma.

It is easy to verify that $R(k')_r^{\mathrm{MP}}$ maps surjectively to $T(k')_r^{\mathrm{MP}}$. By Prop. 4.7.2, $\big(R(k')_r^{\mathrm{MP}}\big)^\Gamma = R(k)_r^{\mathrm{MP}}$ and $\big(T(k')_r^{\mathrm{MP}}\big)^\Gamma = T(k)_r^{\mathrm{MP}}$, where $\Gamma = \operatorname{Gal}(k'/k)$. Our claim is now clear because Γ has order prime to p and the kernel of $R(k')_r^{\mathrm{MP}} \to T(k')_r^{\mathrm{MP}}$ is a pro-p-group in the sense of [PY, before §1]. ∎

§5 The minimal congruent filtration

In this section, we study the smallest filtration which is both admissible and congruent. It turns out that this filtration is also connected, and has other desirable properties.

5.1 **Definition** For a torus T over k, we define a family of smooth models $\{\underline{T}_r^{\mathrm{mc}}\}_{r\geq 0}$ as follows. Let $\underline{T}_0^{\mathrm{mc}}$ be the connected Néron model. For $0 < r < 1$, let $T(\tilde{k})_r^{\mathrm{mc}}$ be the subgroup generated by $\Gamma(\pi, \underline{T}_0^{\mathrm{mc}} \otimes \tilde{\mathcal{O}})$ and the images of $R(\tilde{k})_r^{\mathrm{MP}} \to T(\tilde{k})$ for all morphisms $R \to T$ such that R is induced. There is a smooth model $\underline{T}_r^{\mathrm{mc}}/\mathcal{O}$ such that $\underline{T}_r^{\mathrm{mc}}(\tilde{\mathcal{O}}) = T(\tilde{k})_r^{\mathrm{mc}}$ (just take the dilatation on $\underline{T}_0^{\mathrm{mc}}$ of the closed subgroup $T(\tilde{k})_r^{\mathrm{mc}}/\,\Gamma(\pi, \underline{T}_0^{\mathrm{mc}} \otimes \tilde{\mathcal{O}})$ in $\underline{T}_0^{\mathrm{mc}}(\kappa)$, the closedness is a consequence of [Sp, 2.2.6]).

For $r \geq 1$, we write $r = n + r_0$, with $n \in \mathbb{Z}$, $r \in [0, 1)$. Then we put $\underline{T}_r^{\mathrm{mc}} = (\underline{T}_{r_0}^{\mathrm{mc}})^{(n)}$, the n-th congruence subgroup model (see 2.8).

Finally, we put $T(k)_r^{\mathrm{mc}} = \underline{T}_r^{\mathrm{mc}}(\mathcal{O})$.

5.2 **Proposition** *The filtration $T/k \mapsto \{T(k)_r^{\mathrm{mc}}\}_{r\geq 0}$ is admissible, schematic, connected, and congruent.*

PROOF. The conditions **S1** and **CG** are obvious, and **CN** follows from [Sp, 2.2.6] and a consideration of Greenberg functors. By Lemma 4.4.1, it remains to show that the filtration is admissible. The conditions **F0**, **F2**, and **F4** are also obvious.

Checking **F3**: It is clear that $T(k) \to T'(k)$ maps $\underline{T}_0^{\mathrm{mc}}(\mathcal{O})$ to $(\underline{T}')_0^{\mathrm{mc}}(\mathcal{O})$, and $\Gamma(\pi, \underline{T}_0^{\mathrm{mc}})$ to $\Gamma(\pi, (\underline{T}')_0^{\mathrm{mc}})$. It follows that $T(\tilde{k})_r \subset T'(\tilde{k})_r$ for $r \in [0, 1)$, hence (by Prop. 2.3) there is a canonical morphism $\underline{T}_r^{\mathrm{mc}} \to (\underline{T}')_r^{\mathrm{mc}}$ extending $T \to T'$ for each $r \in [0, 1)$. This shows that $\Gamma(\pi^n, \underline{T}_r^{\mathrm{mc}})$ is mapped into $\Gamma(\pi^n, (\underline{T}')_r^{\mathrm{mc}})$ for all $r \in [0, 1)$, $n \in \mathbb{Z}_{\geq 0}$. **F3** is now proved.

Checking **F1**: Let R be an induced torus such that there is an embedding $T \hookrightarrow R$. Then **F3** implies that $T(k)_r^{\mathrm{mc}} \subset R(k)_r^{\mathrm{MP}}$. Therefore, $T(k)_r^{\mathrm{mc}} \subset R(k)_r^{\mathrm{MP}} \cap T(k)_0^{\mathrm{mc}} = T(k)_r^{\mathrm{MP}}$. This verifies the first statement of **F1**. The second statement, about $T^{\mathrm{mc}}(k)_0$, is obvious. For the third statement, about $T^{\mathrm{mc}}(k)_{0+}$, we need the following lemma.

Lemma *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of tori over k. Then* $\underline{B}^{\mathrm{NR}}(\tilde{\mathcal{O}}) \to \underline{C}^{\mathrm{NR}}(\tilde{\mathcal{O}})$ *is surjective.*

PROOF. By the fact that $H^1(\tilde{k}, A) = 0$, we know that $B(\tilde{k}) \to C(\tilde{k})$ is surjective. The lemma now follows by looking at the Greenberg functors. See [BLR, 9.6, Lemma 2]. ∎

We now return to the proof of **F1**. Let $R \to T$ be a surjective morphism such that R is an induced torus and $\ker(R \to T)$ is a torus. By the above lemma, $G = (\underline{R}^{\mathrm{NR}})_\kappa \to H = (\underline{T}^{\mathrm{NR}})_\kappa$ is surjective. By [Sp, 3.1.1], $R_{\mathrm{u}}(G)$ maps surjectively to $R_{\mathrm{u}}(H)$. Since $R_{\mathrm{u}}(G)$ is the image of $R(\tilde{k})_{0+}^{\mathrm{MP}}$ and $R_{\mathrm{u}}(H)$ is the image of $T(\tilde{k})_{0+}^{\mathrm{MP}}$, **F1** is proved. ∎

5.3 **Lie algebra filtration** Let $\mathfrak{t} = \mathrm{Lie}\, T$. For any $r \geq 0$, let $\mathfrak{t}_r^{\mathrm{mc}} = \mathrm{Lie}\, \underline{T}_r^{\mathrm{mc}}$. Clearly, $\mathfrak{t}_{r+n}^{\mathrm{mc}} = \pi^n \mathfrak{t}_r^{\mathrm{mc}}$ for all $n \geq 0$, $r \geq 0$, and this formula holds for all $n \in \mathbb{Z}$, $r \in \mathbb{R}$ and a unique filtration $\{\mathfrak{t}_r^{\mathrm{mc}}\}_{r \in \mathbb{R}}$ extending $\{\mathfrak{t}_r^{\mathrm{mc}}\}_{r \geq 0}$. It follows that there is a unique *additive norm* (in the sense of [T, 2.9]) $\varphi : \mathfrak{t} \to \mathbb{R} \cup \{+\infty\}$ such that $\mathfrak{t}_r^{\mathrm{mc}} = \{t \in \mathfrak{t} : \varphi(t) \geq r\}$.

To ease the notation, *for the rest of this section, we will omit the superscript* mc *in* $T(k)_r^{\mathrm{mc}}$, $\mathfrak{t}_r^{\mathrm{mc}}$, *or* $\underline{T}_r^{\mathrm{mc}}$.

5.4 Let $r \geq 0$. By Prop. 2.3, there is a natural morphism $\phi : \underline{T}_{r+} \to \underline{T}_r$. Consider $\phi_\kappa : (\underline{T}_{r+})_\kappa \to (\underline{T}_r)_\kappa$. Let H be the image of ϕ_κ with induced reduced structure. The $(\underline{T}_r)_\kappa / H$ is an algebraic group over κ whose $\tilde{\kappa}$-point is $T(\tilde{k})_r / T(\tilde{k})_{r+}$. We will denote this algebraic group by $[T_r / T_{r+}]_\kappa$.

The Lie algebra $\mathrm{Lie}\, \underline{T}_r$ maps surjectively to $\mathrm{Lie}(\underline{T}_r)_\kappa$, and the latter maps surjectively to $\mathrm{Lie}[T_r / T_{r+}]_\kappa$.

Proposition *The kernel of* $\mathrm{Lie}\, \underline{T}_r \to \mathrm{Lie}[T_r / T_{r+}]_\kappa$ *is* $\mathrm{Lie}\, \underline{T}_{r+}$. *Therefore, there is a canonical isomorphism*

$$\mathrm{Lie}[T_r / T_{r+}]_\kappa \simeq \mathfrak{t}_r / \mathfrak{t}_{r+}.$$

PROOF. The Lie algebra of $(\underline{T}_r)_\kappa/H$ is $(\mathfrak{t}_r)\otimes\kappa/\operatorname{Lie} H$. The morphism ϕ_κ is separable if and only if $\operatorname{Lie}(\underline{T}_{r+})_\kappa \to \operatorname{Lie} H$ is surjective. We conclude that $\dim_\kappa[T_r/T_{r+}] \leq \dim_\kappa \mathfrak{t}_r/\mathfrak{t}_{r+}$. The equality holds if and only if ϕ_κ is separable, in which case there is a natural isomorphism $\operatorname{Lie}[T_r/T_{r+}]_\kappa \simeq \mathfrak{t}_t/\mathfrak{t}_{r+}$.

However, we have

$$\sum_{r\in[0,1)} \dim_\kappa[T_r/T_{r+}]_\kappa = \dim T = \sum_{r\in[0,1)} \dim_\kappa \mathfrak{t}_r/\mathfrak{t}_{r+}.$$

Notice that these sums contain only finitely many non-zero terms by S3′. It follows that we must have $\dim_\kappa[T_r/T_{r+}]_\kappa = \dim_\kappa \mathfrak{t}_r/\mathfrak{t}_{r+}$ for all r. ■

5.5 **Proposition** *For any $r > 0$, $[T_r/T_{r+}]_\kappa$ is a vector group.*

PROOF. Clearly, $[T_r/T_{r+}]_\kappa$ is connected, commutative, and unipotent. By [Sp, 3.4.7], the proposition is automatic when the characteristic p of κ is zero, and when $p > 0$, it suffices to show that p kills $[T_r/T_{r+}]_\kappa$.

Assume $p > 0$. By [Sp, 2.2.6] and S3′, there is an induced torus R and a morphism $R \to T$ such that $R(\tilde{k})_r^{\mathrm{MP}}$ maps surjectively to $T(\tilde{k})_r/\Gamma(\pi, \underline{T}_0 \otimes \tilde{\mathcal{O}})$ for all $r \in [0, 1)$. It is easy to check directly that $R(\tilde{k})_r^{\mathrm{MP}}/R(\tilde{k})_{r+}^{\mathrm{MP}}$ is killed by p for all r. It follows immediately that $[T_r/T_{r+}]_\kappa(\tilde{\kappa})$ is killed by p for all $r \in (0, 1)$.

On the other hand, for integer $n \geq 1$, $T(\tilde{k})_n/T(\tilde{k})_{n+1} \simeq \pi^n\tilde{\mathfrak{t}}_0/\pi^{n+1}\tilde{\mathfrak{t}}_0$ is obviously killed by p, where $\tilde{\mathfrak{t}}_0 = \mathfrak{t}_0 \otimes \tilde{\mathcal{O}}$. It follows that $[T_r/T_{r+}]_\kappa(\tilde{\kappa})$ is killed by p for all $r \geq 1$. ■

5.6 **Corollary** *If $r \geq 1$, there is a functorial isomorphism*

$$T(k)_r/T(k)_{r+} \simeq \mathfrak{t}_r/\mathfrak{t}_{r+}.$$

In general, for any $r > 0$, any torus T, $T(k)_r/T(k)_{r+}$ and $\mathfrak{t}_r/\mathfrak{t}_{r+}$ are isomorphic as abstract groups.

PROOF. The statement for $r \geq 1$ follows easily from the proof of Lemma 2.8. The statement for $r > 0$ follows from the fact that for a vector group H over a perfect field κ, we always have $H(\kappa) \simeq \operatorname{Lie} H$ as abstract groups. ■

The above isomorphism is also valid when $(-)^{\mathrm{mc}}$ is replaced by $(-)^{\mathrm{MP}}$, if the torus in question satisifes condition (**T**). It is known as the Moy-Prasad isomorphism. But in general there is no such isomorphism for the original Moy-Prasad filtration.

§6 Filtrations of tori and root subgroups

In this section, we assume that G is quasi-split and use the notations introduced in 1.4.

6.1 **Good filtrations** We refer to [BT1, 6.4.38] for the conditions for a reasonable filtration on $U_0(k) = T(k)$. They call this notion "prolongation of valuation of root datum". These conditions are reproduced in [SS], before I.2.4. There, they call this notion a "good filtration". We will use the terminology of [SS].

Lemma *Let* $T/k \mapsto \{T(k)_r^\dagger\}$ *be admissible. Put* $H_r = T(k)_0^\dagger$ *for* $r \leq 0$, $H_r = T(k)_r^\dagger$ *for* $r \geq 0$. *Then* $\{H_r\}_{r\in\mathbb{R}}$ *is a good filtration.*

PROOF. There are four conditions listed in [SS]. In fact the first condition may not be satisfied as stated in [SS]. This minor point is due to the fact that [SS] works (implicitly) with 1.4.1. Since we work with 1.4.2, the first condition should be modified accordingly. The modified condition is just our definition of H_r for $r \leq 0$.

The only non-obvious condition is the third one: $H_{[r]} \subset H_r \subset H_{(r)}$ (see [SS] for the definitions of $H_{[r]}$ and $H_{(r)}$). Since $\{T(k)_r^{\mathrm{MP}}\}$ is a good filtration (by [SS, the proof of I.2.6]) and $H_r \subset T(k)_r^{\mathrm{MP}}$ by **F1**, it is clear that $H_r \subset H_{(r)}$. It remains to show that $H_{[r]} \subset H_r$.

Let $\tilde{G}$ be the simply connected cover of the derived group of G. Let $\tilde{T}$ be the inverse image of T in $\tilde{G}$. Let $\tilde{H}_{[r]}$, $\tilde{H}_r$ be subgroups of $\tilde{T}(k)$ analogous to $H_{[r]}$ and H_r. From the definition of $H_{[r]}$, it is easy to see that $\tilde{H}_{[r]}$ is mapped *onto* $H_{[r]}$ under $\tilde{G}(k) \to G(k)$.

By [BT2, 4.4.16], $\tilde{T}$ is induced, and hence $\tilde{H}_r = \tilde{T}(k)_r^{\mathrm{MP}}$ by **F2**. It follows that $\{\tilde{H}_r\}$ is a good filtration by the proof of [SS, I.2.6] (for $\tilde{G}$, there is no difference between the two choices of 1.4.1 and 1.4.2). In particular, $\tilde{H}_{[r]} \subset \tilde{H}_r$.

It follows that $H_{[r]}$, the image of $\tilde{H}_{[r]}$, is contained in the image of $\tilde{H}_r$, which is contained in H_r by the functoriality condition **F3**. The lemma is proved. ∎

6.2 **Integral models of the root subgroups** To fix the idea, let us assume also that $S \otimes_k \tilde{k}$ is a maximal $\tilde{k}$-split torus of $G \otimes_k \tilde{k}$. Then, letting $\tilde{k}$ play the role of k in 1.4, we can talk about $U_a(\tilde{k})_{x,r}$.

Proposition *For any* $a \in \Phi$, *any* $r \in \mathbb{R}$, *there exists a unique smooth model scheme* $\underline{U}_{a,x,r}$ *of* U_a *such that* $\underline{U}_{a,x,r}(\tilde{\mathcal{O}}) = U_a(\tilde{k})_{x,r}$. *Moreover, for any* $a \in \Phi$, *we have*

(i) $\underline{U}_{a,x,r}$ *is connected and* $(\underline{U}_{a,x,r})_\kappa$ *is unipotent.*

(ii) *The congruence subgroup* $\Gamma(\pi^n, \underline{U}_{a,x,r})$ *is equal to* $U_a(k)_{x,n+r}$ *for any integer* $n \geq 0$.

For any $a \in \Phi$ *such that* $2a \in \Phi$, *and* $r, s \in \mathbb{R}$ *such that* $2r \geq s$, *there exists a unique smooth model scheme* $\underline{U}_{a,x,r,s}$ *of* U_a *such that* $\underline{U}_{a,x,r,s}(\tilde{\mathcal{O}}) = U_a(\tilde{k})_{x,r}U_{2a}(\tilde{k})_{x,s}$. *Moreover,*

(i) $\underline{U}_{a,x,r,s}$ *is connected and* $(\underline{U}_{a,x,r})_\kappa$ *is unipotent.*

(ii) *The congruence subgroup* $\Gamma(\pi^n, \underline{U}_{a,x,r,s})$ *is equal to* $U_a(k)_{x,r+n,s+n}$ *for any integer* $n \geq 0$.

We refer to [BT2, 4.3] for the proof. We remark that though the detail is a bit long, the idea is quite simple: up to Weil restriction, U_a is just an additive group or a 3-dimensional variant. Once $U_a(k)_{x,r}$ is described explicitly, it is quite straightforward to actually write down the model. Then the stated properties are all easily verified.

Definition For any $a \in \Phi$, let $\mathfrak{u}_a = \operatorname{Lie} U_a$, and we define $\mathfrak{u}_{a,x,r} := \operatorname{Lie} \underline{U}_{a,x,r}$ for all $r \in \mathbb{R}$.

§7 Smooth models for parahoric subgroups

7.1 In this section, we assume that k is strictly henselian. Therefore, G is quasi-split and we adopt the notations in 6.2. By convention, we set $U_0(k)_{x,0} = T(k)_0^{\mathrm{MP}}$, $U_0(k)_{x,0+} = T(k)_{x,0+}^{\mathrm{MP}}$ for any $x \in \mathcal{B}$. Similarly, $\underline{U}_{0,x,r}$ is $\underline{T}_r^{\mathrm{MP}}$, $r = 0, 0+$.

Let F be a facet on A. The group $U_a(k)_{x,0}$ (resp. $U_a(k)_{x,0+}$) is the same for all $x \in F$. Therefore it will be denoted by $U_a(k)_{F,0}$ (resp. $U_a(k)_{F,0+}$). Similarly, we write $\underline{U}_{a,F,0}$ for $\underline{U}_{a,x,0}$, $x \in F$. Let $G(k)_F$ be the subgroup of $G(k)$ generated by $U_a(k)_{F,0}$ for all $a \in \widehat{\Phi}$ (when $a = 0$, $U_a(k)_{F,0}$ means $U_0(k)_0 = T(k)_0$). By definition, $G(k)_F$ is the parahoric subgroup associated to F.

Theorem *There is a canonical affine smooth group scheme $\underline{G}_F$ over $\mathcal{O}$, with generic fiber G, such that*

(i) *$\underline{G}_F$ is connected with $\underline{G}_F(\mathcal{O}) = G(k)_F$;*
(ii) *for each $a \in \widehat{\Phi}$, the morphism $\underline{U}_{a,F,0} \to \underline{G}_F$ is a closed immersion;*
(iii) *for any system $\Phi^{\mathrm{nd}+}$ of positive roots of Φ^{nd}, the multiplication map*

$$\Big(\prod_{a\in\Phi^{\mathrm{nd}-}} \underline{U}_{a,F,0}\Big) \times \underline{T}_0 \times \Big(\prod_{a\in\Phi^{\mathrm{nd}+}} \underline{U}_{a,F,0}\Big) \to \underline{G}_F,$$

is an open immersion, where the two products $\prod_{a\in\Phi^{\mathrm{nd}\pm}}$ can be taken in any order.

The proof of this theorem occupies the rest of this section. It depends on §6 only in that we use the existence statement in 6.2 and the existence of the Néron-Raynaud model. By Lemma 3.3 (iii), it suffices to establish parts (i) and (iii) of the theorem.

7.2 **Lemma (Part (i) of the theorem)** *There is a unique smooth model $\underline{G}_F/\mathcal{O}$ of G/k such*

$$\underline{G}_F(\mathcal{O}) = G(k)_F.$$

Moreover, $\underline{G}_F$ has connected fibers.

PROOF. Let $x_1, \dots, x_s$ be the vertices of F, $\tilde{x}_i$ a lifting of x_i on the extended building of G/k. Let $K_i = \{g \in G(k) : g.\tilde{x}_i = \tilde{x}_i\}$, $K = K_1 \cap \cdots \cap K_s$. By [BT1, 3.3.4], each K_i is a maximal bounded subgroup, and $[K : G(k)_F]$ is finite (see 1.4.3).

Choose a faithful representation $\rho : G \to \mathrm{GL}(V)$ of G. Then $\rho(K_i)$ is contained in a maximal bounded subgroup K'_i of $\mathrm{GL}(V)(k)$. It is known that K'_i is of the form $\mathrm{GL}(L_i)$ for some lattice $L_i \subset V$. We have $\rho^{-1}(\mathrm{GL}(L_i)) = K_i$ by the maximality of K_i.

Let $\underline{G}_i$ be the schematic closure of $\rho(G)$ in $\underline{\mathrm{GL}}(L_i)$. Then it is clear that $\underline{G}_i$ is a model of G with $\underline{G}_i(\mathcal{O}) = K_i$. Now we can apply 2.10 and 2.9 to obtain a smooth model $\underline{G}'$ of G with $\underline{G}'(\mathcal{O}) = K$.

Let n be an integer ≥ 1. Then $\underline{G}'(\mathcal{O}/\pi^n\mathcal{O})$ is the group of κ-points of an algebraic group over κ, via the theory of Greenberg functors. The same theory tells us that the image of $\underline{U}_{a,0}(\mathcal{O}/\pi^n\mathcal{O})$ in $\underline{G}'(\mathcal{O}/\pi^n\mathcal{O})$ is a connected Zariski closed subgroup. It follows that the image of $G(k)_F$ in $\underline{G}'(\mathcal{O}/\pi^n/\mathcal{O})$ is a connected Zariski closed subgroup of finite index by [Sp, 2.2.6], hence simply the neutral component group of $\underline{G}'(\mathcal{O}/\pi^n\mathcal{O})$.

Let $\underline{G}_F$ be the neutral component of $\underline{G}'$. It is a smooth model of G since $\underline{G}$ is. From the above discussion, $\underline{G}_F(\mathcal{O}) = G(k)_F$. The lemma is proved. ∎

7.3 Let Φ_F be set of those $a \in \Phi$ such that there is an affine root with vector part a and defining a hyperplane containing F. Then Φ_F is a (reduced) root system. Define

$$\Phi_{F,\mathrm{nd}} := \Phi^{\mathrm{nd}} \cap (\Phi_F \cup \frac{1}{2}\Phi_F).$$

Notice that this is not necessarily the same as the set of non-divisible elements in Φ_F (which is just Φ_F itself). We can also describe Φ_F as the set of those a such that $U_a(k)_{F,0} \neq U_a(k)_{F,0+}$.

Let Φ^+ be a system of positive roots in Φ. Then $\Phi_F^+ := \Phi^+ \cap \Phi_F$ is a system of positive roots in Φ_F. Let C be the affine chamber containing F in its closure determined by Φ_F^+. Finally, let $\Phi_{F,\mathrm{nd}}^+ = \Phi_{F,\mathrm{nd}} \cap \Phi^+$.

The definition of $G(k)_F$ can be applied to $F = C$ to give the Iwahori subgroup $G(k)_C$. However, it is more convenient to use the following description of $G(k)_C$: it is the subgroup generated by $U_a(k)_{F,0}$ for all $a \in \Phi_{F,\mathrm{nd}}^+ \cup \{0\}$, and $U_a(k)_{F,0+}$ for all $a \in \Phi^{\mathrm{nd}} \smallsetminus \Phi_{F,\mathrm{nd}}^+$.

Let $\mathcal{G}$ be the subgroup of $G(k)$ generated by $U_a(k)$, $a \in \Phi$, and $\underline{T}^{\mathrm{NR}}$. Let $\mathcal{B} = G(k)_C$, $\mathcal{N} = S(G)(k) \cap \mathcal{G}$. Then $W = \mathcal{N}/(\mathcal{N} \cap \mathcal{B})$ is isomorphic to the affine Weyl group of the affine root system associated to (G, S). The chamber C determines a set of reflections $\mathcal{S}$ which generates $\mathcal{N}/(\mathcal{N} \cap \mathcal{B})$.

7.3.1 **Proposition** *The quadruple* $(\mathcal{G}, \mathcal{B}, \mathcal{N}, \mathcal{S})$ *is a (double) Tits system whose Weyl group is* W*, and whose parabolic subgroups are those* $G(k)_F$*,* F *being facets on* $\mathcal{B}$*.*

This is [BT2, 5.2.12]. This is related to 1.4.2, and has very little to do with any scheme-theoretical result in spite of the appearance of $\underline{T}^{\mathrm{NR}}$. Cf. 6.1.

Recall that the Weyl group W_F of Φ_F can be realized as a subgroup of $W = \mathcal{N}/(\mathcal{N} \cap \mathcal{B})$, namely the subgroup generated by those reflections in $\mathcal{S}$ corresponding to hyperplanes containing F. Let $W_F^{\cdot}$ be a subset of $\mathcal{N}$ such that the map $W_F^{\cdot} \to \mathcal{N}/(\mathcal{N} \cap \mathcal{B})$ is injective with image W_F.

7.3.2 **Lemma** *Let $\bar{\mathcal{B}}$ be the image of $\mathcal{B} \hookrightarrow \underline{G}_F(\mathcal{O}) \to \underline{G}_F(\kappa)$. Then we have $\underline{G}_F(\mathcal{O}) = \mathcal{B} W_F^{\cdot} \mathcal{B}$ and $\underline{G}_F(\kappa) = \bar{\mathcal{B}} \bar{W}_F^{\cdot} \bar{\mathcal{B}}$, where $\bar{W}_F^{\cdot}$ is the image of $W_F^{\cdot}$ in $\underline{G}_F(\kappa)$.*

PROOF. The first statement is Bruhat decomposition, a general property of any Tits system. The second statement follows from the first. ∎

7.4 We notice that the inclusion $\mathcal{B} = \underline{G}_C(\mathcal{O}) \subset \underline{G}_F(\mathcal{O})$ induces a morphism $\underline{G}_C \to \underline{G}_F$ of $\mathcal{O}$-schemes by Prop. 2.3. We can also described $\bar{\mathcal{B}}$ as the image of $\underline{G}_C(\kappa) \to \underline{G}_F(\kappa)$. This shows that $\bar{\mathcal{B}}$ is an algebraic subgroup of $\underline{G}_F(\kappa)$.

Similarly, Prop. 2.3 justifies the following definitions: for $a \in \Phi^{\mathrm{nd}} \cup \{0\}$, we set $d_a = \dim U_a$, $\bar{d}_a = \dim \mathrm{Im}(\underline{U}_{a,F,0}(\kappa) \to \underline{G}_F(\kappa))$, $\bar{d}_a^+ = \dim \mathrm{Im}(\underline{U}_{a,F,0+}(\kappa) \to \underline{G}_F(\kappa))$.

7.4.1 **Lemma** *The following inequality holds:*

$$\dim \bar{\mathcal{B}} \leq \sum_{a \in \Phi^+_{F,\mathrm{nd}} \cup \{0\}} \bar{d}_a + \sum_{a \in \Phi^{\mathrm{nd}} \smallsetminus \Phi^+_{F,\mathrm{nd}}} \bar{d}_a^+.$$

PROOF. Let $\bar{U}_a = \mathrm{Im}(\underline{U}_{a,F,0}(\kappa) \to \underline{G}_F(\kappa))$ if $a \in \Phi^+_{F,\mathrm{nd}} \cup \{0\}$, $\bar{U}_a = \mathrm{Im}(\underline{U}_{a,F,0+}(\kappa) \to \underline{G}_F(\kappa))$ if $a \in \Phi^{\mathrm{nd}} \smallsetminus \Phi^+_{F,\mathrm{nd}}$. The map $\bar{U}_a \to \bar{\mathcal{B}}$ is a morphism of algebraic groups over κ by Prop. 2.3. The product morphism $\prod_{a \in \Phi^{\mathrm{nd}} \cup \{0\}} \bar{U}_a \to \bar{\mathcal{B}}$ is surjective by [BT1, 6.4.9]. Hence the lemma. ∎

7.4.2 **Lemma** *For any $w \in W_F^{\cdot}$,*

$$\dim(\bar{\mathcal{B}} \bar{w} \bar{\mathcal{B}} / \bar{\mathcal{B}}) \leq \sum_{a \in \Psi_w} (\bar{d}_a - \bar{d}_a^+),$$

where $\Psi_w = \Phi^+_{F,\mathrm{nd}} \cap (w^{-1}.\Phi^-_{F,\mathrm{nd}})$.

PROOF. We have $\dim \bar{\mathcal{B}} \bar{w} \bar{\mathcal{B}} / \bar{\mathcal{B}} \simeq \bar{\mathcal{B}} / (\bar{\mathcal{B}} \cap \bar{w} \bar{\mathcal{B}} \bar{w}^{-1})$. There is also an obvious surjection $\mathcal{B}/(\mathcal{B} \cap w\mathcal{B}w^{-1}) \to \bar{\mathcal{B}} / (\bar{\mathcal{B}} \cap \bar{w} \bar{\mathcal{B}} \bar{w}^{-1})$. Moreover, by [BT1, 6.4.48], the product map

$$\prod_{a \in \Phi^{\mathrm{nd}} \cup \{0\}} H_a \to \mathcal{B}/(\mathcal{B} \cap w\mathcal{B}w^{-1})$$

is surjective, where

$$H_a = \frac{\mathcal{B} \cap U_a(k)}{\mathcal{B} \cap w\mathcal{B}w^{-1} \cap U_a(k)}.$$

This implies:

(i) the map

$$\prod_{a \in \Phi^{\mathrm{nd}} \cup \{0\}} H_a \to \bar{\mathcal{B}} / (\bar{\mathcal{B}} \cap \bar{w} \bar{\mathcal{B}} \bar{w}^{-1})$$

is surjective.

We now make three more claims for each $a \in \Phi^{\mathrm{nd}} \cup \{0\}$:

(ii) H_a carry the structure of an algebraic group over κ.

(iii) $H_a \to \bar{\mathcal{B}}/(\bar{\mathcal{B}} \cap \bar{w}\bar{\mathcal{B}}\bar{w}^{-1})$ is a morphism of algebraic groups over κ.

(iv) If $a \notin \Psi_w$, H_a is trivial; otherwise the image of H_a in $\bar{\mathcal{B}}/(\bar{\mathcal{B}} \cap \bar{w}\bar{\mathcal{B}}\bar{w}^{-1})$ is of dimension at most $\bar{d}_a - \bar{d}_a^+$.

It is clear that the lemma follows from (i)–(iv). We now prove claims (ii), (iii), and (iv).

(ii): Clearly, H_a is non-trivial only when both $a \in \Phi_{F,\mathrm{nd}}^+$ and $w.a \notin \Phi_{F,\mathrm{nd}}^+$ hold, that is, when $a \in \Psi_w$. Assume that $a \in \Psi_w$, then $H_a = U_a(k)_{F,0}/U_a(k)_{F,0+}$. In this case, Prop. 6.2 shows that

$$H_a \simeq \frac{U_a(k)_{F,0}/\,\Gamma(\pi, \underline{U}_{a,F,0})}{U_a(k)_{F,0+}/\,\Gamma(\pi, \underline{U}_{a,F,0})}$$

is the quotient of two algebraic subgroups of $(\underline{U}_{a,F,0})_\kappa$, hence is an algebraic group over κ.

(iii) follows from the fact that the map $\bar{U}_a \to \bar{\mathcal{B}}$ is a morphism of algebraic groups over κ (see the proof of the Lemma 7.4.1).

(iv): We have seen that H_a is trivial if $a \notin \Psi_w$. Assume that $a \in \Psi_w$. Then $H_a \to \bar{\mathcal{B}}/(\bar{\mathcal{B}} \cap \bar{w}\bar{\mathcal{B}}\bar{w}^{-1})$ factors through a morphism

$$H_a \to \frac{\mathrm{Im}(\underline{U}_{a,F,0}(\kappa) \to \underline{G}_F(\kappa))}{\mathrm{Im}(\underline{U}_{a,F,0+}(\kappa) \to \underline{G}_F(\kappa))}.$$

Thus (iv) is obvious. ∎

7.4.3 **Lemma** *Let $w \in W_F^\cdot$ be such that its image in W_F is the longest element of W_F, then $\bar{\mathcal{B}}\bar{w}\bar{\mathcal{B}}$ is dense in $\underline{G}_F(\kappa)$. Consequently, $(\bar{w}^{-1}\bar{\mathcal{B}}\bar{w})\bar{\mathcal{B}}$ is also dense in $\underline{G}_F(\kappa)$.*

PROOF. We first recall that the action of w exchanges $\Phi_{F,\mathrm{nd}}^+$ and $\Phi_{F,\mathrm{nd}}^-$. This also shows that $\sum_{a\in\Phi_{F,\mathrm{nd}}^+}(\bar{d}_a - \bar{d}_a^+) = \sum_{a\in\Phi_{F,\mathrm{nd}}^-}(\bar{d}_a - \bar{d}_a^+)$.

Lemma 7.3.2 exhibits $\underline{G}_F(\kappa)$ as a union of finitely many constructible sets. At least one of them have to be dense. Hence at least one of $\bar{\mathcal{B}}\bar{w}'\bar{\mathcal{B}}$ has to be of dimension equal to $\dim G = \sum_{a\in\Phi^{\mathrm{nd}}\cup\{0\}} d_a$. By the preceding two lemmas, the dimension is $\le \sum_{a\in\Phi^{\mathrm{nd}}\cup\{0\}} \bar{d}_a$. Since $d_a \ge \bar{d}_a$, this implies that $d_a = \bar{d}_a$ for all $a \in \Phi^{\mathrm{nd}} \cup \{0\}$.

Thus $\underline{U}_{a,F,0}(\kappa) \to \underline{G}_F(\kappa)$ has finite kernel. By Prop. 6.2, $\bar{d}_a > \bar{d}_a^+$ for all $a \in \Phi_{F,\mathrm{nd}}$. By Lemmas 7.4.1 and 7.4.2, in order that $\dim \bar{\mathcal{B}}\bar{w}'\bar{\mathcal{B}} = \dim G$, we must have $\Phi_{F,\mathrm{nd}}^+ = w'^{-1}.\Phi_{F,\mathrm{nd}}^-$. This happens precisely when $w = w'$. ∎

7.5 **Lemma (Part (iii) of the theorem)** *The product morphism*

$$\Big(\prod_{a\in\Phi^{\mathrm{nd}-}} \underline{U}_{a,F,0}\Big) \times \underline{T}_0 \times \Big(\prod_{a\in\Phi^{\mathrm{nd}+}} \underline{U}_{a,F,0}\Big) \to \underline{G}_F$$

is an open immersion. Here, the two products $\prod_{a\in\Phi^{\mathrm{nd}\pm}}$ can be taken in any order.

PROOF. Let w be as in the preceding lemma, and $\mathcal{B}' = w^{-1}\mathcal{B}w$. By the preceding lemma, the image of $\mathcal{B}'\mathcal{B}$ in $\underline{G}_F(\kappa)$ is Zariski dense.

By [BT1, 6.4.9], $\mathcal{B} = \mathcal{B}_- T(k)_0 \mathcal{B}_+$, where $\mathcal{B}_- = \prod_{a\in\Phi^{\mathrm{nd}-}} U_a(k)_{C,0}$, $\mathcal{B}_+ = \prod_{a\in\Phi^{\mathrm{nd}+}} U_a(k)_{C,0}$. Similarly, there is a decomposition $\mathcal{B}' = \mathcal{B}'_- T(k)_0 \mathcal{B}'_+$. We have $\mathcal{B}'_+ \subset \mathcal{B}_+$, $\mathcal{B}_- \subset \mathcal{B}'_-$.

Therefore,

$$\mathcal{B}'\mathcal{B} = \mathcal{B}'_- T(k)_0 \mathcal{B}'_+ \mathcal{B} = \mathcal{B}'_- T(k)_0 \mathcal{B} = \mathcal{B}'_- T(k)_0 \mathcal{B}_- T(k)_0 \mathcal{B}_+ = \mathcal{B}'_- T(k)_0 \mathcal{B}_+.$$

That the image of $\mathcal{B}'_- T(k)_0 \mathcal{B}_+ \to \underline{G}_F(\kappa)$ is dense is precisely the same as: the morphism

$$\Big(\prod_{a\in\Phi^{\mathrm{nd}-}} \underline{U}_{a,F,0}\Big) \times \underline{T}_0 \times \Big(\prod_{a\in\Phi^{\mathrm{nd}+}} \underline{U}_{a,F,0}\Big) \to \underline{G}_F$$

induces a dominant map on the special fibers. The lemma now follows from Lemma 3.1. ∎

7.6 Theorem 7.1 is the key result about parahoric subgroups, from which other results follow easily as in [BT2, 4.6.4]. The most useful structural result is a complete description of the isomorphism class of the maximal reductive quotient of $(\underline{G}_F)_\kappa$. Unfortunately, in the literature we can only find a statement [T, 3.5.1] describing the separable isogeny class of $(\underline{G}_F)_\kappa$. For the sake of reference, we record the full statement here.

We first review the notion of coroots for a quasi-split group G relative to a maximal k-split torus S. The coroot of $a \in \Phi(G, S)$ will be an element $a^\vee$ in $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, S)$ such that $\langle a, a^\vee\rangle = 2$. If S is of rank 1, $a^\vee$ is simply defined by this property.

In general, for each $a \in \Phi$ such that $2a \notin \Phi$, there is a separable extension E/k and a k-homomorphism $\phi : \mathrm{Res}_{E/k}(\mathrm{SL}_2) \to G$ pulling back $\{\pm a\}$ to the roots $\{\pm a'\}$ of $\mathrm{Res}_{E/k}(\mathrm{SL}_2)$ (relative to the standard maximal torus of $\mathrm{Res}_{E/k}(\mathrm{SL}_2)$). We now define the coroot of a to be the push-forward of $(\phi^* a)^\vee$ by ϕ. Similarly, if $a, 2a \in \Phi$, there is a separable extension E/k and a k-homomorphism $\phi : \mathrm{Res}_{E/k}\,\mathrm{SU}(3) \to G$, where $\mathrm{SU}(3)$ is a quasi-split special unitary group over E in 3 variables relative to a suitable quadratic extension of E. The homomorphism ϕ sends $\{\pm a, \pm 2a\}$ to the roots $\{\pm a', \pm 2a'\}$ of $\mathrm{Res}_{E/k}\,U(3)$. We then define the coroot of a to be the push-forward of $(\phi^* a)^\vee$ by ϕ. Notice that $(\phi^* a)^\vee$ is already defined in both cases because the k-rank of $\mathrm{Res}_{E/k}\,\mathrm{SL}(2)$ or $\mathrm{Res}_{E/k}\,\mathrm{SU}(3)$ is 1.

Now we resume to the discussion of $\underline{G}_F$.

Proposition *The image $\bar{S}$ of the closed immersion $(\underline{S}^{\mathrm{NR}})_\kappa \to (\underline{G}_F)_\kappa$ is a maximal torus of $(\underline{G}_F)_\kappa$. The image of $\bar{S}$ in the maximal reductive quotient $(\underline{G}_F)_\kappa^{\mathrm{red}}$ of $(\underline{G}_F)_\kappa$ is a maximal torus there. The character group and co-character group of $\bar{S}$ are canonically identified with those of S. Under these identifications, the root datum of $\big((\underline{G}_F)_\kappa^{\mathrm{red}}, \bar{S}\big)$ is $(X^*(S), \Phi_F, X_*(S), \Phi_F^\vee)$, where $\Phi_F^\vee = \{a^\vee : a \in \Phi_F\}$ and the bijection $a \mapsto a^\vee$ in the definition of root datum is the obvious one.*

For example, this allows us to see the isomorphism class of $(\underline{G}_F)_\kappa^{\mathrm{red}}$ by pure thought, when $\underline{G}$ is split adjoint or split simply connected, and F is a vertex. We refer to [GY] for examples.

§8 Smooth models associated to concave functions

In this section, we continue to assume that k is strictly henselian. Therefore, G is quasi-split and we adopt the notations in 6.2. Let $T/k \mapsto \{T(k)^{\dagger}_r\}_{r\geq 0}$ be a schematic admissible filtration. By Lemma 6.1, $\{T(k)^{\dagger}_r\}$ gives a good filtration on the centralizer of S in $G(k)$. To simplify the notation, we will often omit the superscrpt † in this section. By convention, we set $U_0(k)_{x,r} = T(k)_r$, $\underline{U}_{0,x,r} = \underline{T}_r$ for all $r \geq 0$, $x \in \mathcal{B}$.

8.1 **The condition (T)** Recall that T, the centralizer of S in G, is a maximal torus. We recall also that when T satisfies condition (T), the filtration $\{T(k)^{\dagger}_r\}_{r\geq 0}$ on $T(k)$ is independent of the choice of the admissible filtration, and can be computed easily (see Lemma 4.7.4). Although we do not need to assume (T) for the main theorem, it is useful to note that (T) holds under one of the following conditions:

(i) G is simply connected.
(ii) G is adjoint.
(iii) G is split over a tamely ramified extension.

Indeed, under conditions (i) or (ii), T is induced by [BT2, 4.4.16]. It is clear that (iii) implies (T).

8.2 **Concave functions** We recall from [BT1, 6.4] the following definition: A function $f : \widehat{\Phi} \to \tilde{\mathbb{R}}$ is called *concave* if for any non-empty finite family $\{a_i\}_{i=1}^{s}$ of elements in $\widehat{\Phi}$ such that $\sum_{i=1}^{s} a_i \in \widehat{\Phi}$, we have

$$\sum_{i=1}^{s} f(a_i) \geq f\Big(\sum_{i=1}^{s} a_i\Big).$$

From now on, we fix a point $x \in \mathcal{B}(G,k)$. For any concave $f : \widehat{\Phi} \to \tilde{\mathbb{R}}$, let $G(k)_{x,f}$ be the subgroup of $G(k)$ generated by $U_a(k)_{x,f(a)}$ for all $a \in \widehat{\Phi}$. If $a \in \widehat{\Phi}$ is such that $2a \notin \Phi$, we let $\underline{U}_{a,x,f} = \underline{U}_{a,x,f(a)}$. If $a \in \Phi$ is such that $2a$ is also in Φ, we let $\underline{U}_{a,x,f} = \underline{U}_{a,x,f(a),f(2a)}$ (see 6.2 for the notation).

8.3 **Theorem** *Let* $f : \widehat{\Phi} \to \tilde{\mathbb{R}}_{\geq 0} \smallsetminus \{\infty\}$ *be concave. Then*

(i) *There is a unique smooth model* $\underline{G}_{x,f}$ *of* G *such that* $\underline{G}_{x,f}(\mathcal{O}) = G(k)_{x,f}$.
(ii) *For each* $a \in \widehat{\Phi}^{\mathrm{nd}}$*, the schematic closure of* U_a *in* $\underline{G}_{x,f}$ *is* $\underline{U}_{a,x,f}$.
(iii) *The multiplication morphism*

$$\Big(\prod_{a\in\Phi^{\mathrm{nd}-}} \underline{U}_{a,x,f}\Big) \times \underline{T}_{f(0)} \times \Big(\prod_{a\in\Phi^{\mathrm{nd}+}} \underline{U}_{a,x,f}\Big) \to \underline{G}_{x,f}$$

is an open immersion, and induces an isomorphism on the special fiber if $f(0) > 0$*. Here, the two products* $\prod_{a\in\Phi^{\mathrm{nd}\pm}}$ *can be taken in any order.*

8.3.1 **Lemma** *For any concave function* $h : \widehat{\Phi} \to \tilde{\mathbb{R}}_{>0} \smallsetminus \{\infty\}$*, the image of the multiplication map*

$$\Big(\prod_{a\in\Phi^{\mathrm{nd}-}} \underline{U}_{a,x,f}(\mathcal{O})\Big) \times \underline{U}_{0,x,f}(\mathcal{O}) \times \Big(\prod_{a\in\Phi^{\mathrm{nd}+}} \underline{U}_{a,x,f}(\mathcal{O})\Big) \to G(k)_{x,f}$$

is dense in the π*-adic topology.*

Proof. Bruhat-Tits [BT1, 6.4.48] shows that this map is a bijection when k is complete. The argument there proves this lemma. Notice that it is a consequence of the theorem that this map is a bijection in general. ∎

For each real number $t \geq 0$, let $f_t : \widehat{\Phi} \to \tilde{\mathbb{R}}$ be the function defined by $f_t(a) = tf(a)$. It is clear that f_t is concave for any t. By **S3**, we can find a finite sequence $0 = t_0 < t_1 < \cdots < t_n$ such that $f_{t_n} = f_1 = f$, and $f_{t_{i+1}} \leq f_{t_i} + 1$ for $i = 0, \ldots, n-1$.

We now use induction to prove part (i) of the theorem for $f = f_{t_i}$. The case $i = 0$ is just Theorem 7.1. Assume that the theorem is established for $f = f_{t_i}$.

We will apply Lemma 3.4.1 with $\mathcal{K}' = \{\underline{U}_{a,x,f_{t_i}}(\mathcal{O})\}_{a\in\widehat{\Phi}^{\mathrm{nd}}}$, $\mathcal{K}'' = \{\underline{U}_{a,x,f_{t_{i+1}}}\}_{a\in\widehat{\Phi}^{\mathrm{nd}}}$. Let $H(\kappa)$ be as defined before Lemma 3.4.1.

8.3.2 **Lemma** $H(\kappa)$ *is Zariski closed.*

Proof. If $f(0) > 0$, then by Lemma 8.3.1, $H(\kappa)$ is isomorphic to the direct product of $\mathrm{Im}(\underline{U}_{a,x,f_{t_{i+1}}}(\kappa) \to \underline{G}_{x,f_{t_i}}(\kappa))$. This is obviously a locally closed subset of $\underline{G}_{x,f_{t_i}}$. By [Sp, 2.2.4 (ii)], it is actually closed.

Suppose $f(0) = 0$. Let V be the image of the product morphism $\prod_a \underline{U}_{a,x,f_{t_{i+1}}}(\kappa) \to \underline{G}_{x,f_{t_i}}(\kappa)$. By [BT1, 6.4.7, 6.4.9 (iii)], there exist finitely many elements $n_1, \ldots, n_r$ (namely, coset representatives of $T(k)_0\backslash N_{f_{t_{i+1}}}$, where $N_{f_{t_{i+1}}}$ is as defined in [BT1, 6.4.2]) such that $H(\kappa) = Vn_1 \cup \cdots \cup Vn_r$. Again, it follows from [Sp, 2.2.4] that $H(\kappa)$ is closed. ∎

The preceding lemma, Lemmas 3.5, condition **S2**, and Prop. 6.2 shows that the hypothesis of Lemma 3.4.1 is satisfied. It follows that part (i) of the theorem holds for $f = f_{t_{i+1}}$.

8.3.3 Since (iii) implies (ii) by Lemma 3.3 (iii), it remains to prove (iii). We first remark that if $f(0) > 0$ and if the schematic filtration we are using is connected, then (iii) is an immediate consequence of Lemma 3.3 (ii) and Lemma 8.3.1

In general, we again need to use induction on i. Assume the validity of the theorem for $f = f_{t_i}$. We will apply Lemma 3.4.2 to establish the theorem for $f = f_{t_{i+1}}$. We adapt the notations used in the proof of Lemma 8.3.2. We also notice that the argument there shows that

$\dim H = \dim V$, and V is open in H. Let $\underline{U}$ be the "big cell" for $\underline{G}_{x,f_{t_i}}$ as in Lemma 3.4.2. We need to show that $H(\kappa) \cap \underline{U}(\kappa) = V(\kappa)$.

We first handle the case $f(0) > 0$. In this case, $H(\kappa) = V(\kappa) \subset \underline{U}(\kappa)$ by Lemma 8.3.1.

Next, assume $f(0) = 0$. In this case, $\underline{U}_{0,x,0}(\kappa)$ is connected. By [Sp, 2.2.6], H is connected.

We can write $\underline{U}(\kappa) = AB$, where $A = \prod_{a \in \Phi^{\mathrm{nd}-}} \underline{U}_{a,x,f_{t_i}}(\kappa)$, $B = \prod_{a \in \{0\} \cup \Phi^{\mathrm{nd}+}} \underline{U}_{a,x,f_{t_i}}(\kappa)$. Similarly, $V(\kappa) = A'B'$, where $A' = A \cap V(\kappa)$, $B' = B \cap V(\kappa)$. Then A, B, A', B' are subgroups by [BT1, 6.4.9 (ii)]. Now if $g \in H(\kappa) \cap \underline{U}(\kappa)$ but $g \notin V(\kappa)$, we can write $g = ab$, $a \in A$, $b \in B$. It follows that $AabB \subset H(\kappa)$. Clearly, $AabB$ contains an open subset of dimension equal to the dimension of H, but is disjoint from V. This is not possible because H is connected and hence two non-empty open subsets must overlap.

This contradiction shows that $H(\kappa) \cap \underline{U}(\kappa) = V(\kappa)$. The theorem for $f = f_{t_{i+1}}$ now follows from Lemma 3.4.2. The proof of the main theorem is now complete. ∎

8.4 **Corollary** *If $r = f(0) > 0$, and $(\underline{T}_r)_\kappa$ is connected, then $(\underline{G}_{x,f})_\kappa$ is connected unipotent.*

PROOF. By (iii) of the theorem, $(\underline{G}_{x,f})_\kappa$ is isomorphic to $\mathbb{A}^n_\kappa$, $n = \dim G$. ∎

8.5 **Corollary**

$$\operatorname{Lie} \underline{G}_{x,f} = \bigoplus_{a \in \widehat{\Phi}^{\mathrm{nd}}} \operatorname{Lie} \underline{U}_{a,x,f}.$$

PROOF. $\operatorname{Lie} \underline{G}_{x,f}$ is the tangent space to $\underline{G}_{x,f}$ at the identify section. Since the identity section factors through the open sub-scheme $\prod_{a \in \widehat{\Phi}^{\mathrm{nd}}} \underline{U}_{a,x,f}$, the tangent space can be computed there. ∎

8.6 We recall that the Moy-Prasad group $G(k)_{x,r}$ is $G(k)_{x,f}$, where f is such that $f(a) = r$ for all $r \in \widehat{\Phi}$.

Corollary *For any $r \geq 0$,*

- (i) *There is a unique smooth model $\underline{G}_{x,r}$ of G such that $\underline{G}_{x,r}(\mathcal{O}) = G(k)_{x,r}$.*
- (ii) *For each $a \in \hat{\Phi}$, the schematic closure $\underline{U}_a$ of U_a in $\underline{G}_{x,r}$ is $\underline{U}_{a,x,r}$ when $2a \notin \Phi$, and is $\underline{U}_{a,x,r,r}$ when $2a \in \Phi$.*
- (iii) *The multiplication morphism $\prod_{a \in \widehat{\Phi}^{\mathrm{nd}}} \underline{U}_a \to \underline{G}_{x,r}$ is an open immersion. It induces an isomorphism on the special fibers when $r > 0$. Here the product is taken in an order as specified in Theorem 8.3.*

8.7 We now recall the definition of Moy-Prasad filtration on $\mathfrak{g} = \operatorname{Lie} G$: for $a = 0$, let $\mathfrak{u}_0 = \mathfrak{t}$, $\mathfrak{u}_{0,x,r} = \mathfrak{t}_r^{\mathrm{MP}}$; then $\mathfrak{g}_{x,r} = \sum_{a \in \widehat{\Phi}} \mathfrak{u}_{a,x,r}$. Notice that this is defined in a way similar to the definition of $G(k)_{x,r}$, but there is no relation between the two definitions a priori. Our theory now gives a natural connection:

Corollary *Assume that* $T(k)_r = T(k)_r^{\mathrm{mc}} = T(k)_r^{\mathrm{MP}}$. *For any* $r \geq 0$, $\operatorname{Lie} \underline{G}_{x,r} = \mathfrak{g}_{x,r}$.

If we redefine $\mathfrak{u}_{0,x,r}$ to be $\mathfrak{t}_r^{\mathrm{mc}}$, and redefine $\mathfrak{g}_{x,r}$ accordingly, then $\operatorname{Lie} \underline{G}_{x,r} = \mathfrak{g}_{x,r}$ whenever $T(k)_r = T(k)_r^{\mathrm{mc}}$.

8.8 **Corollary** *Assume that the schematic filtration* $T/k \mapsto \{T(k)_r^{\dagger}\}_{r \geq 0}$ *is congruent. Then*

$$\Gamma(\pi^n, \underline{G}_{x,f}) = G(k)_{f+n}$$

for all integer $n \geq 0$. *In particular,*

$$\Gamma(\pi^n, \underline{G}_{x,r}) = G(k)_{x,r+n}$$

for all $r \geq 0$, *integer* $n \geq 0$.

PROOF. Clear from 2.8, Lemma 3.5 and Prop. 6.2. ∎

§9 Comments and additional results

9.1 **Etale descent** Drop the assumption that k is strictly henselian. If $G(\tilde{k})_{x,f}$ is $\operatorname{Gal}(\tilde{k}/k)$-stable, then $\underline{G \otimes \tilde{k}}_{x,f}$ descends to a group scheme $\underline{G}_{x,f}$ over $\mathcal{O}$. This is always the case for the Moy-Prasad group: for any $x \in \mathcal{B}(G,k)$, $r \geq 0$, there is a unique smooth model $\underline{G}_{x,r}/\mathcal{O}$ of G/k such that $\underline{G}_{x,r}(\mathcal{O}') = G(k')_{x,r}$ for any unramified extension k'/k.

9.2 **Non-positive concave functions** We have assumed $f \geq 0$ for simplicity. But this is not a really a restriction, because of the following:

Lemma *Let* $f : \widehat{\Phi} \to \mathbb{R}$ *be a concave function, then there exists* $v \in X_*(S) \otimes \mathbb{R}$ *such that* $f(a) + a(v) \geq 0$ *for all* $a \in \Phi \cup \{0\}$.

Since $G(k)_{x,f} = G(k)_{x+v,h}$, where $h(a) = f(a) + a(v)$, this lemma reduces the case of an arbitrary $\mathbb{R}$-valued f to the case $f \geq 0$. Because we are dealing with discrete valuations, using $\tilde{\mathbb{R}}$ is only a matter of convenience: the collection of subgroups $G(k)_{x,f}$ arising from $\tilde{\mathbb{R}}$-valued concave f's is the same as that of subgroups arising from $\mathbb{R}$-valued f's.

PROOF. (G. Prasad) Let $V = X^*(S) \otimes \mathbb{R}$ so that $\Phi \subset V$. Let $\Phi' = \{(a, f(a)) : a \in \widehat{\Phi}\} \subset V' := V \oplus \mathbb{R}$. Let C be the convex hull of Φ'. We notice that the definition of a concave function implies that $f(0) \geq 0$. This, together with [BT1, 6.4.6], show that C does not contain any point of the form $(0, t)$ with $t < 0$.

It follows (e.g. by Farkas' lemma, or by using Theorem 3.4 (b) from W. Rudin, *Functional analysis*) that there is a hyperplane passing through $(0, 0)$ separating $(0, -1)$ and C: there exists $\varphi \in \mathrm{Hom}(V', \mathbb{R})$ such that $\varphi(0, -1) < 0$ and $\varphi(C) \geq 0$. Write $\varphi = (w, b) \in \mathrm{Hom}(V, \mathbb{R}) \oplus \mathbb{R}$. Then the conditions are $b > 0$ and $a(w) + bf(a) \geq 0$ for all $a \in \widehat{\Phi}$. We can now conclude the proof of the lemma by taking $v = b^{-1}w$. ∎

9.3 **Stabilizer** *In this subsection, apartments and facets refer to those on the enlarged building, and k is strictly henselian.* Let Ω be a bounded subset of $A(S)$, the apartment associated to S. Define

$$\mathrm{Fix}(G(k), \Omega) = \{g \in G(k) : g.x = x \text{ for all } x \in X\}.$$

In [BT2, 4.6.18], Bruhat-Tits constructed canonical smooth model for this group from the existence of certain $\underline{G}_{x, f_\Omega}$ and a lemma in algebraic geometry. Here, we will give a different and simple argument. Let F be a facet, we first show that there is a canonical model for

$$\mathrm{Stab}(G(k), F) = \{g \in G(k) : g.F = F\}.$$

9.3.1 **Lemma** $\mathrm{Stab}(G, F)$ *is an intersection of finitely many maximal bounded subgroups of* $G(k)$.

PROOF. Let C be a chamber containing F in its closure, X the set of vertices of C. There is a homomorphism $G(k) \to \mathrm{Aut}(X)$ by [T, 2.5]. Let Ξ be its image Let Y be the set of vertices of F, and Ξ_F be the subgroup of Ξ leaving Y invariant.

Let $Y_1, \ldots, Y_s$ be the orbits for the action of Ξ_F on Y. Then it is clear that

$$\mathrm{Stab}(G(k), F) = \mathrm{Stab}(G(k), Y_1) \cap \cdots \cap \mathrm{Stab}(G(k), Y_s).$$

By applying [BT1, 3.3.4] to 1.4.1, each $\mathrm{Stab}(G(k), Y_i)$ is a maximal bounded subgroup. The lemma is proved. ∎

9.3.2 **Lemma** *There is a unique smooth model* $\underline{\mathrm{Stab}}(G, F)$ *of* G *such that* $\underline{\mathrm{Stab}}(G, F)(\mathcal{O}) = \mathrm{Stab}(G(k), F)$. *Moreover,* $\underline{G}_F$ *is the neutral component of* $\underline{\mathrm{Stab}}(G, F)$.

PROOF. The first statement follows from the preceding lemma and from the proof of Lemma 7.2.

Consider the canonical morphism $\underline{G}_F \to \underline{\mathrm{Stab}}(G, F)$ given by Prop. 2.3. By 1.4.3, $\underline{G}_F(\mathcal{O})$ is of finite index in $\underline{\mathrm{Stab}}(G, F)(\mathcal{O})$. Again by looking at the associated Greenberg functors and arguing as in the proof of Lemma 7.2, we see that $\underline{G}_F$ is the neutral component of $\underline{\mathrm{Stab}}(G, F)$. ∎

9.3.3 **Lemma** *For any $x \in F$, there is a unique smooth model $\underline{\mathrm{Fix}}(G, x)$ of G such that $\underline{\mathrm{Fix}}(G, x)(\mathcal{O}) = \mathrm{Fix}(G(k), x)$. Moreover, $\underline{G}_F$ is the neutral component of $\underline{\mathrm{Fix}}(G, x)$.*

PROOF. By 1.4.3, $G(k)_{x,0} \subset \mathrm{Fix}(G(k), x) \subset \mathrm{Stab}(G(k), F)$, and the indices of these inclusions are finite. It follows easily that $\underline{\mathrm{Fix}}(G, x)$ can be obtained from $\underline{\mathrm{Stab}}(G, F)$ by performing a dilatations. The second statement is proved by the argument for Lemma 9.3.2. ∎

9.3.4 **Lemma** *For any bounded set Ω on $A(S)$, there is a unique smooth model $\underline{\mathrm{Fix}}(G, \Omega)$ of G such that $\underline{\mathrm{Fix}}(G, \Omega)(\mathcal{O}) = \mathrm{Fix}(G(k), \Omega)$. Moreover, $\underline{G}_{x, f_\Omega}$ is the neutral component of $\underline{\mathrm{Fix}}(G(k), \Omega)$, where x is any point on $A(S)$ and f_Ω is as in [BT1, 6.4.2].*

PROOF. Clearly, $\mathrm{Fix}(G(k), \Omega) = \bigcap_{x \in \Omega} \mathrm{Fix}(G(k), x)$. In fact, there are only finitely many distinct $\mathrm{Fix}(G(k), x)$ in this intersection. Therefore, the existence of $\underline{\mathrm{Fix}}(G, \Omega)$ follows from the preceding lemma, 2.10, and 2.9. The second statement is proved by the argument for Lemma 9.3.2. ∎

9.4 **Remark on Schneider-Stuhler theory** The group $U_F^{(e)}$ used in the theory of Schneider-Stuhler [SS] is the same as the Moy-Prasad group $G(k)_{x,e+}$, where x is any point in the facet F, and $e \in \mathbb{Z}_{\geq 0}$ (see [V]). If we replace the Moy-Prasad filtration for a torus in the definition of $U_F^{(e)}$ by the minimal congruent filtration, we get another collection of groups, which we will denote by $U_F^{[e]}$.

Everything in Schneider-Stuhler theory remains valid if we replace $U_F^{(e)}$ by $U_F^{[e]}$, with the same proof. The advantage of using $U_F^{[e]}$ over $U_F^{(e)}$ is that $U_F^{[e]}$ has a very compact and conceptual definition:

> *Let $\underline{U}_F = \underline{\mathrm{Stab}}(G, F)$, and let $\underline{U}_F^{+}$ be the dilatation on $\underline{U}_F$ of $R_{\mathrm{u}}\big((\underline{U}_F)_\kappa^\circ\big)$. Then $U_F^{[e]} = \Gamma(\pi^e, \underline{U}_F^{+})$ for all $e \in \mathbb{Z}_{\geq 0}$.*

It is clear that $\underline{U}_F^{(e)}$ is invariant under any automorphism of $\underline{U}_F$. In particular, it is normalized by $P_F^\dagger$ (see [SS, before I.2, and before I.2.7]). In contrast, the definition of $U_F^{(e)}$ is rather lengthy. However, to develop Schneider-Stuhler theory using $U_F^{[e]}$, one still needs to go through most of the work needed to define $U_F^{(e)}$, e.g. the root group decomposition.

§10 The compact groups in the construction of tame types

The compact groups ${}^\circ K = \vec{G}(k)_{x,\vec{r}}$ used in [Y] for constructing supercuspidal types and supercuspidal representations are formed using the theory of $G(k')_{x,f}$ for a suitable extension k'/k, but they may not be of the form $G(k)_{x,f}$. See 10.3 below.

Therefore, the theory developed here does not apply to ${}^\circ K$ directly. However, it is still easy to show that ${}^\circ K$ underlies a smooth model. The point is that in the context of [Y], everything is assumed to split over a tamely ramified extension.

10.1 We refer to [Y, sections 1 and 2] for the notation and construction of $\vec{G}(k)_{x,\vec{r}}$. Here, we will first consider a more general situation. Assume that $k'/\tilde{k}$ is a tamely ramified finite extension with Galois group Γ. Let $x \in \mathcal{B}(G,k)$. We use the inclusions $\mathcal{B}(G,k) \subset \mathcal{B}(G,\tilde{k}) \subset \mathcal{B}(G,k')$ to regard x as points on $\mathcal{B}(G,\tilde{k})$ and $\mathcal{B}(G,k')$.

Let S be a maximal k'-split torus in $G \otimes k'$ such that $x \in A(G \otimes k', S)$, and T the centralizer of S in $G \otimes k'$. Let f be a concave function on $\Phi(G \otimes k', S) \cup \{0\}$. Then we can consider $G(k')_{x,f}$.

The extension k'/k is always Galois. We assume that $G(k')_{x,f}$ is $\mathrm{Gal}(k'/k)$-stable and put

$$G(k)_{x,f} = G(k')_{x,f}^{\mathrm{Gal}(k'/k)}, \qquad G(\tilde{k})_{x,f} = G(k')_{x,f}^{\Gamma}.$$

Notice that the group $G(k)_{x,f}$ certainly depends on the choice of (k', S), although this is only coded in the fact that f is a function on $\Phi(G \otimes k', S) \cup \{0\}$.

It is easy to see that the group $\vec{G}(k)_{x,\vec{r}}$ considered in [Y, sections 1 and 2] is the same as $G(k)_{x,f}$ for some suitable (k', S, f).

***10.2* Proposition** *Assume that* $G(k')_{x,f}$ *is* $\mathrm{Gal}(k'/k)$*-stable. There is a unique smooth scheme* $\underline{G}_{x,f}$ *over* $\mathcal{O}$*, which is a model of* G*, such that* $\underline{G}_{x,f}(\tilde{\mathcal{O}}) = G(\tilde{k})_{x,f}$.

PROOF. By étale descent, we may and do assume that $k = \tilde{k}$ is strictly henselian.

Let $\mathcal{O}'$ be the ring of integers in k', and $\underline{G}'/\mathcal{O}'$ be the model of $G \otimes k'$ associated to x, f by the main theorem. Let $\underline{G}'' = \mathrm{Res}_{\mathcal{O}'/\mathcal{O}}(\underline{G}_{x,f})$. This is a smooth model of $\mathrm{Res}_{k'/k}(G \otimes k')$ by Lemma 2.5. The group Γ acts on $\underline{G}''/\mathcal{O}$ naturally by Prop. 2.3. By definition, $\underline{G}''(\mathcal{O})^{\Gamma} = G(k)_{x,f}$.

We now define $\underline{G}_{x,f}$ to be $(\underline{G}'')^{\Gamma}$. By Prop. 2.11, $\underline{G}_{x,f}$ is a smooth model of G with the desired set of integral points. ∎

***10.3* Example** Assume that 2 is invertible in $\mathcal{O}$. Let $G = \mathrm{GL}_2$. Consider a 2-step twisted Levi sequence (G^0, G^1) in G, with $G^1 = G$,

$$G^0(R) = \left\{ \begin{pmatrix} a & b \\ \pi b & a \end{pmatrix} \mid a^2 - \pi b^2 \in R^{\times} \right\}, \qquad \text{so that } G^0 \simeq \mathrm{Res}_{E/k}(\mathbb{G}_{\mathrm{m}}),\ E = k(\sqrt{\pi}).$$

Let $\vec{r} = (0,2)$, and let x be the middle point of the chamber whose corresponding Iwahori subgroup is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathcal{O},\ ad - bc \in \mathcal{O}^{\times},\ \pi | c \right\}.$$

Then we can consider $\vec{G}(k)_{x,\vec{r}}$ and $\vec{G}(\tilde{k})_{x,\vec{r}}$. By the preceding proposition, there is a unique smooth model $\underline{G}$ of G such that $\underline{G}(\tilde{\mathcal{O}}) = \vec{G}(\tilde{k})_{x,\vec{r}}$. In fact, it is easy to calculate $\underline{G}$ explicitly: $G = \mathrm{Spec}\, k[a,b,c,d,\delta]/((ad-bc)\delta - 1)$, and

$$\underline{G} = \mathrm{Spec}\, \mathcal{O}[a, b, \pi^{-2}(d-a), \pi^{-3}(c - \pi b)].$$

It is then easy to check that the maximal torus of $(\underline{G})_\kappa$ is of rank 1. It follows that $\vec{G}(k)_{x,\vec{r}}$ is not of the form $G(k)_{y,f}$ for any (y, S, f), because the maximal torus of $(\underline{G}_{y,f})_\kappa$ is always of rank 2 (when $f(0) = 0$) or 0 (when $f(0) > 0$).

10.4 Let $\underline{G}$ be the canonical model of G associated to $\vec{G}_{x,\vec{r}}$ by the preceding proposition, where $\vec{G} = (G^0, \ldots, G^d)$ is a tamely ramified twisted Levi sequence, and $\vec{r}$ is an increasing sequence of non-negative real numbers. We now discuss the structure of $\underline{G}_\kappa$. For simplicity and without loss of generality, we assume that κ is strictly henselian for a while.

We recall that $\underline{G}(\mathcal{O})$ is of the form

$$\underline{G}(\mathcal{O}) = J^0 J^1 \cdots J^d,$$

where $J^0 = \underline{G}^0(\mathcal{O})$, $\underline{G}^0 = \underline{\mathrm{Fix}}(G^0, x)$, and J^i are as defined in [Y], just before section 4. By the the preceding proposition, J^i underlies a smooth model $\underline{J}^i$ of G^i for $i = 1, \ldots, d$.

Proposition *Let H_i be the image of J^i in $\underline{G}(\kappa)$.*

(i) *The groups $H_0, \ldots, H_d$ are Zariski closed in $\underline{G}(\kappa)$.*

(ii) *For any $i \geq 1$, H_i is connected unipotent, and $H_i \cdots H_d$ is a Zariski closed, connected, unipotent, and normal subgroup in $\underline{G}_\kappa$.*

(iii) *The neutral component $\underline{G}_\kappa^\circ$ of $\underline{G}_\kappa$ is*

$$H_0^\circ H_1 \cdots H_d,$$

where $H_0^\circ = \underline{G}_x^0(\kappa)$ is the neutral component of $\underline{G}^0(\kappa)$.

(iv) *The canonical morphism $\underline{G}_x^0 \to \underline{G}$ of $\mathcal{O}$-group schemes induces an isomorphism from $(\underline{G}_x^0)_\kappa^{\mathrm{red}}$ to $(\underline{G}_\kappa^\circ)^{\mathrm{red}}$, the maximal reductive quotient of $\underline{G}_\kappa^\circ$.*

PROOF. (i) There is a morphism $\underline{J}^i \to \underline{G}$ by Prop. 2.3. Since H_i is the image of $\underline{J}^i(\kappa) \to \underline{G}(\kappa)$, it is closed by [Sp, 2.2.5 (ii)].

(ii) follows from Corollary 8.4 and Lemma 4.7.2 and the fact that $J^i \cdots J^d$ is normal in $\underline{G}(\mathcal{O})$ (see [Y, sections 3 and 4]).

(iii) is obvious: $H_0^\circ H_1 \cdots H_d$ is connected of finite index in $\underline{G}(\kappa)$.

(iv) Let R be the unipotent radical of H_0°. Then $R' = RH_1 \cdots H_d$ is connected unipotent, and normal in $\underline{G}^\circ(\kappa)$. We have $\underline{G}^\circ(\kappa)/R' \simeq H_0^\circ/(H_0^\circ \cap R')$, since $H_0^\circ \cap R' \supset R$ and H_0°/R has no unipotent normal subgroup, we must have $H_0^\circ \cap R' = R$. Now (iv) is clear. ∎

10.5 Now assume that k is locally compact as in [Y]. That is, k is complete and κ is finite. Suppose that we are given a generic datum as in [Y, section 15]. Then we can construct a representation ${}^\circ\rho$ of $\underline{G}(\mathcal{O})$ from the datum so that $(\underline{G}(\mathcal{O}), {}^\circ\rho)$ is a type [Y, Corollary 15.3]. Here we give a preliminary discussion about the shape of ${}^\circ\rho$. By [Y, section 4], ${}^\circ\rho$ is a tensor product $\tau_0 \otimes \cdots \otimes \tau_d$, where $\tau_0, \ldots, \tau_d$ are representations of $\underline{G}(\mathcal{O})$. Moreover,

(i) τ_0 is inflated from a representation of the finite group $(\underline{G}_\kappa^{\mathrm{red}})(\kappa)$ which contains a cuspidal representation when restricted on $(\underline{G}_\kappa^\circ)^{\mathrm{red}}(\kappa)$.

(ii) τ_d is a linear character.

(iii) For $i = 1, \ldots, d-1$, τ_i is trivial on $J^{i+1} \cdots J^d$. Therefore, it is determined by $\tau_i | J^i$ and $\tau_i | J^0 \cdots J^{i-1}$.

(iv) There is a quotient of $\underline{J}^i(\kappa)$ which is a Heisenberg group. The representation $\tau_i | J^i$ is inflated from a Heisenberg representation of that quotient.

(v) By 10.4, there is a canonical smooth model $\underline{G}^{i-1}$ of G^{i-1} such that $\underline{G}^{i-1}(\mathcal{O}) = J^0 \cdots J^{i-1}$. There is a linear character ϕ_{i-1} of $\underline{G}^{i-1}(\mathcal{O})$ such that $\tau_i' = \left(\tau_i | \underline{G}^{i-1}(\mathcal{O})\right) \otimes \phi_{i-1}^{-1}$ is inflated from a representation of $(\underline{G}_\kappa^{i-1})^{\mathrm{red}}(\kappa)$. The latter is essentially a Weil representation.

Therefore, up to some linear characters, all the ingredient representations are on groups of the form $\underline{H}(\mathcal{O})$, where $\underline{H}$ is a smooth group scheme over $\mathcal{O}$, and the representations are inflated from $\underline{H}(\kappa)$. These results suggest that algebraic geometry and group schemes should play an important role in the representation theory of p-adic groups.

References

[BLR] S. Bosch, W. Lütkebohmert, and M. Raynaud: *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete **21**, Springer Verlag (1990).

[BT1] F. Bruhat et J. Tits: *Groupes réductifs sur un corps local, Chapitre I*, Publ. Math. I.H.E.S. **41**, 5–251 (1972).

[BT2] F. Bruhat et J. Tits: *Groupes réductifs sur un corps local, Chapitre II*, Publ. Math. I.H.E.S. **60**, 197–376 (1984).

[CY] C.-L. Chai and J.-K. Yu: *Congruences of Neron model for tori and the Artin conductor*, Annals of Math. **154**, 347–382 (2001).

[D] S. DeBacker: *Some applications of Bruhat-Tits theory to Harmonic analysis on a reductive p-adic group*, Michigan Math. J., to appear

[E] B. Edixhoven: *Néron models and tame ramification*, Compositio Math. **81**, 291–306 (1992).

[ELL] B. Edixhoven, Q. Liu, and D. Lorenzini: *The p-part of the group of components of a Néron model*, J. Alg. Geom. **5**, 801–813 (1996).

[GHY] W.T. Gan, J. Hanke and J.-K. Yu: *On an exact mass formula of Shimura*, Duke Math. J. **107**, 103–133 (2001).

[GY] W.T. Gan and J.-K. Yu: *Schémas en groupes et immeubles des groupes exceptionelles sur un corps local, Deuxieme partie: Le groupes F_4 et E_6*, Bulletin Math. Soc. France, to appear.

[L] E. Landvogt: *A compactification of the Bruhat-Tits building*, Springer-Verlag (1995).

[MP1] A. Moy and G. Prasad: *Unrefined minimal K-types for p-adic groups*, Inv. Math. **116**, 393–408 (1994)

[MP2] A. Moy and G. Prasad: *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helvetici **71**, 96–121 (1996)

[PR] G. Prasad and M.S. Raghunathan: *Topological central extensions of semi-simple groups over local fields*, Annals of Math. **119**, 143–268 (1984).

[PY] G. Prasad and J.-K. Yu: *On finite group actions on reductive groups and buildings*, Invent. Math. **147**, 545–560 (2002).

[SS] P. Schneider and U. Stuhler: *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. **85**, 97–191 (1997).

[Se] J.-P. Serre: *Local fields*, Springer-Verlag (1979)

[Sp] T.A. Springer: *Linear algebraic groups*, Second Ed., Birkhauser (1998).

[St] R. Steinberg: *Lectures on Chevalley groups*, Yale University (1967).

[T] J. Tits: *Reductive groups over local fields*, Automorphic forms, representations, and L-functions (A. Borel and W. Casselman, Eds.), v.1., 29–69 (1977).

[Y] J.-K. Yu: *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14**, 579–622 (2001).

[V] M.F-. Vignéras: *Cohomology of sheaves on the building and R-representations*, Invent. Math. **127**, 349–373 (1997).