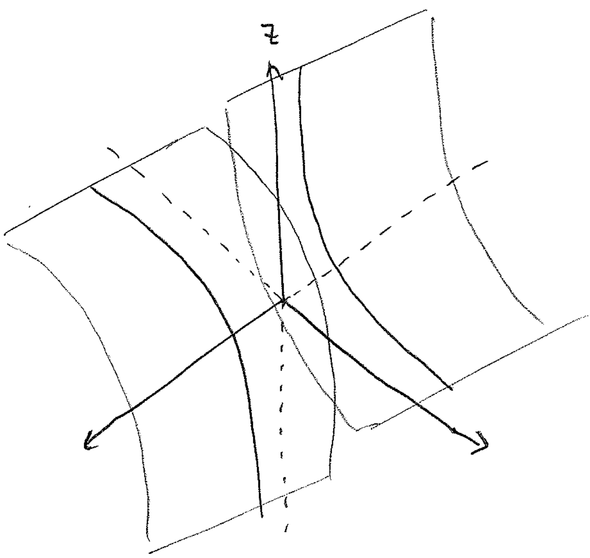
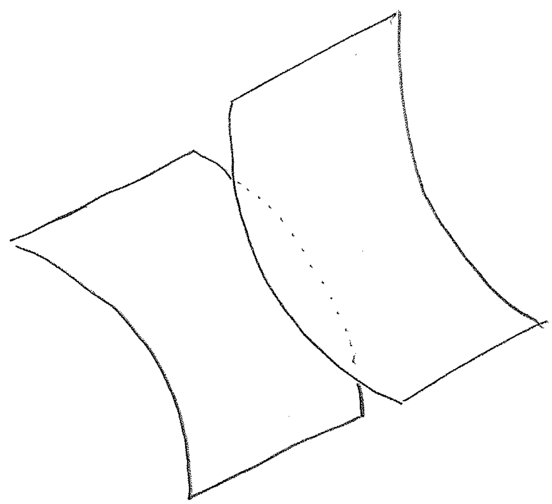
Midterm 1
MA 271, Fall 2009
Instructor: Javier Zuniga

NAME: ______________________________

No calculators or notes allowed. There are a total of 100 points. Read the exam carefully. Show your work for full credit.

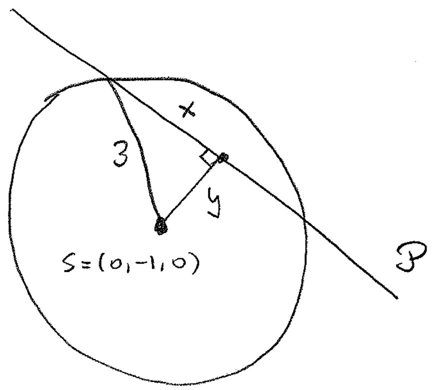
1. (10 points) Sketch and name the surface $yz = 1$





hyperbolic cylinder

2. (10 points) The plane $2x + y + 2z = 4$ intersects the sphere $x^2 + (y+1)^2 + z^2 = 9$ at a circle. Find the radious of that circle.



$$y = d(S, \mathcal{P}) = \frac{|\overrightarrow{P_0S} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{|\langle 0,-1,-2\rangle \cdot \langle 2,1,2\rangle|}{|\langle 2,1,2\rangle|} = \frac{|-1-4|}{3} = \frac{5}{3}$$

$S = (0,-1,0)$

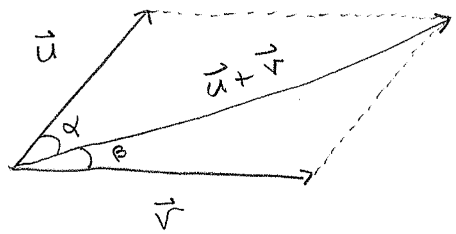
$P_0 = (0,0,2)$

$\vec{n} = \langle 2,1,2\rangle$

$$x = \sqrt{9 - y^2} = \frac{2}{3}\sqrt{14}$$

3. Prove the following statements using vectors.

(a) (5 points) Show that if a parallelogram has all its sides of equal length then the diagonals bisec the angles of the parallelogram.



If $|\vec{u}| = |\vec{v}|$ we neet to show $\cos\alpha = \cos\beta$

$$\cos\alpha = \frac{\vec{u}\cdot(\vec{u}+\vec{v})}{|\vec{u}|\,|\vec{u}+\vec{v}|} = \frac{\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{u}+\vec{v}|} = \frac{\vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{v}}{|\vec{v}|\,|\vec{u}+\vec{v}|}$$

$$= \frac{\vec{v}\cdot(\vec{u}+\vec{v})}{|\vec{v}|\,|\vec{u}+\vec{v}|} = \cos\beta$$

becouse $|\vec{u}| = |\vec{v}|$ and $\vec{u}\cdot\vec{u} = |\vec{u}|^2 = |\vec{v}|^2 = \vec{v}\cdot\vec{v}$

(b) (5 points) Show that $|\vec{v}|\vec{u} + |\vec{u}|\vec{v}$ and $|\vec{v}|\vec{u} - |\vec{u}|\vec{v}$ are orthogonal.

$$(|\vec{v}|\vec{u} + |\vec{u}|\vec{v})\cdot(|\vec{v}|\vec{u} - |\vec{u}|\vec{v}) = |\vec{v}|\vec{u}\cdot(|\vec{v}|\vec{u} - |\vec{u}|\vec{v}) + |\vec{u}|\vec{v}\cdot(|\vec{v}|\vec{u} - |\vec{u}|\vec{v})$$

$$= |\vec{v}|\vec{u}\cdot|\vec{v}|\vec{u} - |\vec{v}|\vec{u}\cdot|\vec{u}|\vec{v} + |\vec{u}|\vec{v}\cdot|\vec{v}|\vec{u} - |\vec{u}|\vec{v}\cdot|\vec{u}|\vec{v}$$

$$= |\vec{v}|^2|\vec{u}|^2 - |\vec{u}||\vec{v}|\,\vec{u}\cdot\vec{v} + |\vec{u}||\vec{v}|\,\vec{u}\cdot\vec{v} - |\vec{u}|^2|\vec{v}|^2 = 0$$

4. (10 points) Find the equation of the plane parallel to the lines $x = 1 + t, y = 2 - t, z = -2t$ and $x = -1 - t, y = 3t, z = 0$ that passes through the origin.

$\vec{v}_1 = \langle 1, -1, -2 \rangle$

$\vec{v}_2 = \langle -1, 3, 0 \rangle$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 3 & 0 \end{vmatrix} = \langle 6, 2, 2 \rangle$$

Therefore:

$$\langle 6, 2, 2 \rangle \cdot \langle x-0, y-0, z-0 \rangle = 0$$

OR

$$3x + y + z = 0$$

5. (20 points) True or False?

(a) The curve $< t^2, t^2, t^2 >$ is smooth for al real numbers. T ~~F~~

(b) The vector function $\vec{r}(t) = |t|\hat{k}$ is continous. ~~T~~ F

(c) The triple product $\vec{u} \times \vec{v} \cdot \vec{u}$ is always zero. ~~T~~ F

(d) The lines $x = 1 + t, y = 2 - t, z = 3 + 2t$ and $x = 1 - t, y = 2 + t, z = 3 - 2t$ are equal.
. ~~T~~ F

(e) The surface $z^2 - x^2 - y^2 = 1$ is a hyperboloid. ~~T~~ F

(f) The curve $< \cos t, 1 - t, \sin t >$ is the intersection of a circular cylinder and a plane.
. T ~~F~~

(g) The curve $< 3\cos t, 4, e^{2t} >$ has non-zero torsion. T ~~F~~

(h) If the acceleration vector of a particle is the zero vector the path described by this particle may still have non-zero curvature. T ~~F~~

(i) If the velocity and acceleration vectors are parallel then the torsion is zero. ~~T~~ F

(j) The area of the parallelogram determined by the vectors $< 1, 0, 0 >$ and $< 0, 0, 1 >$ is $\sqrt{2}$.
. T ~~F~~

6. (15 points) A train engine moves at a constant speed on a straight horizontal track. As the engine moved along a marble was thrown into the air vertically with respect to the engine. The marble, which continue to move with the same forward speed as the engine, rejoined the engine 1 sec after it was fired. Because of the forward motion the marble made an angle of 45 degrees with the horizontal. Find the the speed of the engine.

Travel time $= \dfrac{2|\vec{v}_0|\sin 45^\circ}{g}$ Speed of engine $= |\vec{v}_0|\cos 45^\circ$

$\Rightarrow \quad 1 = \dfrac{2|\vec{v}_0|\sin 45^\circ}{g} \quad \Rightarrow \quad |\vec{v}_0|\cos 45^\circ = \dfrac{g}{2}$ since $\cos 45^\circ = \sin 45^\circ$

7. (10 points) Reparametrize the curve $< e^t \cos t, e^t, e^t \sin t >$ by arc length from the point $t = 0$. Find the point that is a distance of 2 along the curve from the point $(1, 1, 0)$ in the direction opposite to the direction of increasing arc-length.

$$\vec{r}' = \langle e^t\cos t - e^t\sin t,\ e^t,\ e^t\sin t + e^t\cos t \rangle$$

$$|\vec{r}'| = \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t + e^{2t} + e^{2t}\sin^2 t + e^{2t}\cos^2 t} = \sqrt{3}\, e^t$$

$$s = \int_0^t \sqrt{3}\, e^{\tau}\, d\tau = \sqrt{3}\, e^{\tau}\Big|_0^t = \sqrt{3}\,(e^t - 1)$$

$$t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$$

New parametrization:

$$\vec{r}(s) = \left\langle \left(\frac{s}{\sqrt{3}}+1\right)\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right),\ \left(\frac{s}{\sqrt{3}}+1\right),\ \left(\frac{s}{\sqrt{3}}+1\right)\sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right) \right\rangle$$

The required point is achived at $-t = \ln\left(\frac{2}{\sqrt{3}}+1\right)$ therefore

$$\text{Point} = \left\langle \left(\frac{2}{\sqrt{3}}+1\right)^{-1}\cos\left(\ln\left(\frac{2}{\sqrt{3}}+1\right)^{-1}\right),\ \left(\frac{2}{\sqrt{3}}+1\right)^{-1},\ \left(\frac{2}{\sqrt{3}}+1\right)^{-1}\sin\left(\ln\left(\frac{2}{\sqrt{3}}+1\right)^{-1}\right) \right\rangle$$

8. (15 points) Find $\vec{T}$, $\vec{N}$ and the curvature for

$$\vec{r}(t) = < \cosh t, -\sinh t, t >$$

$\cosh t = \frac{e^t + e^{-t}}{2}$

$\sinh t = \frac{e^t - e^{-t}}{2}$

Then find $\vec{B}$ and torsion.

$\vec{v} = \langle \sinh t, -\cosh t, 1 \rangle$

$|\vec{v}|^2 = \frac{(e^t - e^{-t})^2}{4} + \frac{(e^t + e^{-t})^2}{4} + 1 = \frac{e^{2t} - 2 + e^{-2t} + e^{2t} + 2 + e^{-2t}}{4} + 1 = \frac{e^{2t} + e^{-2t}}{2} + \frac{2}{2} = \cosh^2(2t)$

$\vec{T} = \frac{1}{\cosh(2t)} \langle \sinh t, -\cosh t, 1 \rangle$

$\frac{d\vec{T}}{dt} = -\frac{2\sinh(2t)}{\cosh^2(2t)} \langle \sinh(t), -\cosh(t), 1 \rangle + \frac{1}{\cosh(2t)} \langle \cosh(t), -\sinh(t), 0 \rangle$

$\left| \frac{d\vec{T}}{dt} \right| = \frac{1}{\cosh(2t)} - \frac{8\sinh(t)\cosh(t)\sinh(2t)}{\cosh^3(2t)} + \frac{4\sinh^2(2t)\cosh^2(t)}{\cosh^4(2t)}$

$\vec{N} = \frac{d\vec{T}}{dt} \Big/ \left| \frac{d\vec{T}}{dt} \right|$ $\qquad \vec{B} = \vec{T} \times \vec{N}$

For curvature and torsion: $\vec{a} = \langle \cosh t, -\sinh t, 0 \rangle$

$\vec{J} = \langle \sinh t, -\cosh t, 0 \rangle$

$\Rightarrow \quad |\vec{v} \times \vec{a}| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix} = |\langle \sinh t, \cosh t, 1 \rangle| = \cosh(2t)$

$K = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\cosh(2t)}{\cosh^3(2t)} = \frac{1}{\cosh^2(2t)}$

$\tau = \frac{\begin{vmatrix} \vec{v} \\ \vec{a} \\ \vec{J} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \frac{-\cosh^2 t + \sinh^2 t}{\cosh^2(2t)} = -\frac{1}{\cosh(2t)}$