Midterm 2
MA 271, Fall 2009
Instructor: Javier Zuniga NAME: ________________

No calculators or notes allowed. There are a total of 100 points. Read the exam carefully. Show your work for full credit (except for the True and False questions).

1. (10 points) Find the sum of the series

$$\sum_{n=2}^{\infty} \frac{1}{1-n^2}$$

$$\frac{1}{1-n^2} = \frac{1}{(1-n)(1+n)} = -\frac{1}{(n+1)(n-1)}$$

$$\frac{1}{(n+1)(n-1)} = \frac{A}{n-1} + \frac{B}{n+1} \iff 1 = A(n+1) + B(n-1) \iff A = 1/2,\ B = -1/2$$

$$\frac{1}{1-n^2} = -\frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = -\frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}\right)$$ Two telescopic series

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{1-n^2} = -\frac{1}{2}\left(\frac{1}{2-1}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}\cdot\frac{3}{2} = -\frac{3}{4}$$

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2. (5 points) What is the minimum number of terms we need to add from the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2}$$

in order to guarantee an aproximation with an error of less than 0.01?

$$\underbrace{\frac{1}{2^2} - \frac{1}{3^2} + \cdots + \frac{1}{10^2}}_{9 \text{ terms}} - \frac{1}{11^2}$$

$$\text{error} \leq \frac{1}{11^2} < \frac{1}{10^2} = 0.01$$

ANSWER = 9 terms

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3. (10 points) Find the sum of the series

$$\sum_{n=3}^{\infty}(-1)^n\frac{n}{3^n}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{and differentiate:} \quad -\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

Multiply all by x : $\displaystyle -\frac{x}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^n n x^n = -x + 2x^2 + \sum_{n=3}^{\infty}(-1)^n n x^n$

Plug-in for $x=1/3$: $\displaystyle \sum_{n=3}^{\infty}(-1)^n\frac{n}{3^n} = \frac{-1/3}{(1+1/3)^2} + \frac{1}{3} - 2\left(\frac{1}{3}\right)^2 = -\frac{11}{144}$

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4. (20 points) Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n}$ $= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2\sum_{n=1}^{\infty} \frac{1}{n^2+n} \leq 2\sum_{n=1}^{\infty} \frac{1}{n^2}$

Convergent by comparison test and p-test.

(b) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \frac{\tan 1/n}{1/n} = \lim_{x\to 0} \frac{\tan x}{x} \overset{\text{L'Hôspital}}{=} \lim_{x\to 0} \sec^2 x = 1$$

Divergent by limit comparison test with harmonic series

(c) $\sum_{n=1}^{\infty} a_n$ if $a_1 = 2$ and $a_{n+1} = \left(\frac{1+\sin(n)}{n}\right) a_n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+\sin(n)}{n} = 0 \quad \text{because} \quad 0 \leq \frac{|1+\sin(n)|}{n} \leq \frac{2}{n} \to 0$$

Since $0<1$ this is convergent by ratio test

(d) $\sum_{n=1}^{\infty} a_n$ if $a_1 = 1/3$ and $a_{n+1} = a_n^{1/n}$

$$a_n = \left(\frac{1}{3}\right)^{\frac{1}{(n-1)!}} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{3}\right)^{\frac{1}{(n-1)!}} = \left(\frac{1}{3}\right)^0 = 1 \neq 0$$

Divergent by n-term test.

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5. (10 points) True or False?

(a) If $\sum a_n$ is convergent then $\sum \frac{1}{a_n}$ is convergent. T ~~F~~

(b) If $\sum a_n$ is divergent then $\sum n a_n$ is also divergent. ~~T~~ F

(c) The function $f(x,y,z) = |x| y \sin(1/z)$ is differentiable everywhere except at the origin.

· f is not differentiable at $(0,1,1) \neq (0,0,0)$ T ~~F~~

(d) The set $\{(x,y,z) \mid xy > 0,\ (x,y,z) \neq (10,10,10)\}$ is open. ~~T~~ F

(e) The set $\{(x,y) \mid x = y\}$ is closed. ~~T~~ F

(f) The sets [0,1) and $[1,\infty)$ are not open nor closed. In fact $[1,\infty)$ is closed! T ~~F~~

(g) If $f(x,y) = \sum_{n=0}^{100} x^n y^{100-n}$ then $f_{xy} \neq f_{yx}$. T ~~F~~

(h) The function $f(x,y) = |x|/(\sqrt{|x|} + \sqrt{|y|})$ can be made continuos at the origin. ~~T~~ F

(i) The series $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^{1.02}}$ is divergent. T ~~F~~

(j) The sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ is equal to the limit $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$. ~~T~~ F

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6. (10 points) Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that the limit does not exists.

(a) $f(x,y) = \cos^{-1}\left(\frac{x^3-y^3}{x^2+y^2}\right)$ $\quad 0 \leq \frac{|x^3|}{x^2+y^2} = |x| \frac{x^2}{x^2+y^2} \leq |x|$ because $\frac{x^2}{x^2+y^2} \leq 1$

Therefore: $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$ and in a similar way $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$

Thus $\lim_{(x,y)\to(0,0)} \cos^{-1}\left(\frac{x^3-y^3}{x^2+y^2}\right) = \cos^{-1}(0) = \frac{\pi}{2}$

(b) $f(x,y) = \ln\left(\frac{x^2y}{x^3+y^3}\right)$ $\quad$ Take $y = mx$ then

$$\frac{x^2y}{x^3+y^3} = \frac{x^2\,mx}{x^3+m^3x^3} = \frac{x^3\,m}{x^3(1+m^3)} = \frac{m}{1+m^3}$$

The limit doesn't exists!

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7. (15 points) Find the values of x for which the power series converges:

$$1 + \frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{27} + \frac{x^4}{16} + \frac{x^5}{243} + \frac{x^6}{64} + \cdots$$

and compute the sum.

$$= \sum_{n=0}^{\infty} \left(\frac{x^2}{4}\right)^n + \frac{x}{3} \sum_{n=0}^{\infty} \left(\frac{x^2}{9}\right)^n \qquad \left|\frac{x^2}{4}\right| < 1 \Rightarrow |x| < 2$$

$$\left|\frac{x^2}{9}\right| < 1 \Rightarrow |x| < 3$$

for $x = \pm 2$ we have: $\sum_{n=0}^{\infty} \left(\frac{(\pm 2)^2}{4}\right)^n = \sum_{n=0}^{\infty} \left(\frac{4}{4}\right)^n = \sum_{n=0}^{\infty} 1^n = \sum_{n=1}^{\infty} 1 = \infty$

ANS : $|x| < 2$

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8. (20 points) Find the sum of the series

$$\sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^n}{(n+\frac{1}{2})}$$

$$\left(\text{HINT: } \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C\right)$$

$$\left(1+(-x^2)\right)^{-1/2} = 1 + \sum_{n=1}^{\infty} \binom{-1/2}{n} (-x^2)^n = 1 + \sum_{n=1}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}$$

Integration term by term;

$$\text{LHS} = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

$$\text{RHS} = x + \sum_{n=1}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+1}}{(2n+1)}$$

Plug-in for $x=1$:

$$\frac{\pi}{2} = \arcsin(1) = 1 + \sum_{n=1}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{2n+1}$$

$$\Rightarrow \quad \pi - 2 = 2 \sum_{n=1}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{2n+1} = \sum_{n=1}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{n+\frac{1}{2}}$$

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