Midterm 1, Solutions
MA 266
Instructor: Javier NAME: ____________________

No calculators or notes allowed. There are a total of 100 points. Read the exam carefully. Show your work for full credit.

1. (10 points) Determine the order of the given differential equation and state whether it is linear or non-linear.

   (a) $y'' + \tan(t + y) = 1$ order 2, non-linear
   (b) $y' + t(y')^3 = 0$ order 1, non-linear
   (c) $y'''' + y'' + y = t^2 + 1$ order 4, linear
   (d) $(y')^{yy''} + \frac{1}{t^2} = 0$ order 2, non-linear
   (e) $t^t y''' + \frac{y'}{\cos t} = \sqrt{t}$ order 3, linear

2. (20 points) Find the general solution of

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{\sin\frac{y}{x}}$$

   in explicit form. (HINT: use the substitution $v = y/x$)

   Since $y = vx$ then $y' = xv' + v$ and the equation turns into

$$x\frac{dv}{dx} + v = v + \frac{1}{\sin v}$$

   which can be rewritten as $\sin v dv = dx/x$. Integration yields $-\cos v = \ln|x| + C$ or $\cos(y/x) = -\ln|x| - C$. Thus the explicit form of the solutions is

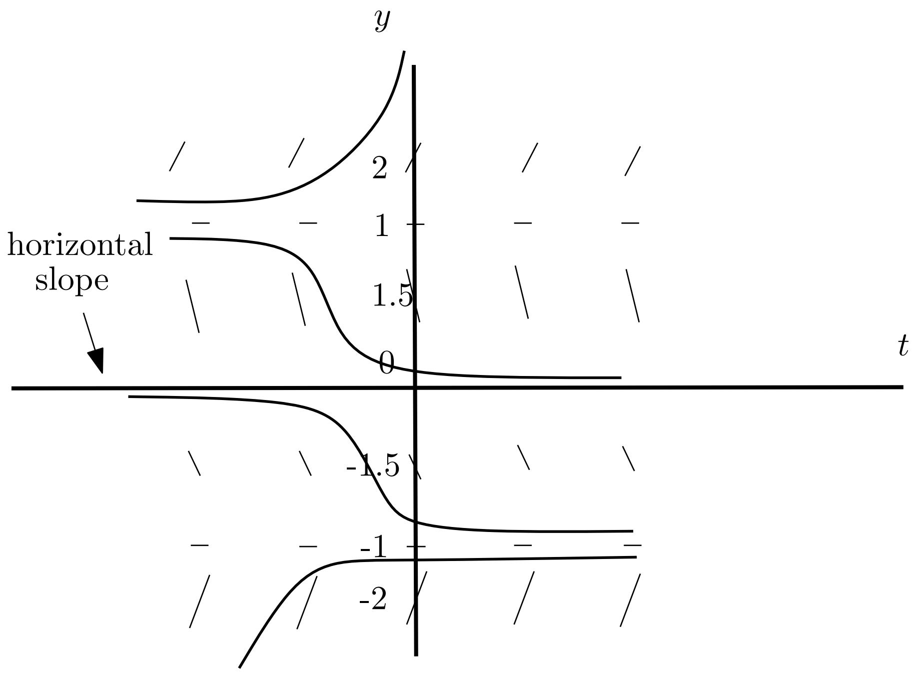
$$y(x) = x \arccos(-\ln|x| - C)$$

3. (10 points) Given the equation

$$y' = y^4 - y^2$$

   (a) Sketch the direction field.

   Notice that this is an autonomous equation and hence it is enough to graph the slopes at the y axis and then repeat them along the t axis. Since $y' = y^4 - y^2 = y^2(y^2 - 1) = y^2(y+1)(y-1)$ it is convinient to graph the slopes at the following points: -2,-1,-1.5,0,1.5,2.



(b) Find the equilibrium solutions and state their stability.
Since $y' = y^4 - y^2 = y^2(y^2-1) = y^2(y+1)(y-1)$ we have the equilbrium solutions at -1, 0, 1 and they are stable, semi-stable and un-stable respectively by looking at the direction field.

4. (20 points) At the start, 5 lbs of salt are dissolved in 20 gal of water. A solution with a concentration of salt of 2 lb/gal is added at a rate of 3 gal/min, and the well-stirred mixture is drained out at the same rate of flow. How long should this process continue in order to raise the amount of salt in the tank to 25 lbs?

The formula we use to model the amount of salt at any given time is $S' = ratein - rateout$ where the rate is the product of the rate of flow and the concentration. For the rate in this is just (3)(2). For the rate out notice that the tank always contains 20 gal of water (because the rate of flow in is the same as the rate of flow out). Thus the rate out is $(S/20)(3)$. Since we have 5 lbs of salt at the beginning the initial value problem can be stated as $S' = 6 - (3S)/(20)$ with $S(0) = 5$.

To solve the IVP we can rewrite the equation as $S' + 3S/20 = 6$ from which we get the integrating factor $\mu(t) = \exp(3t/20)$. Integration then yields

$$S(t) = 40 + Ce^{-3t/20}$$

To find C is enough to plug-in the initial condition $S(0) = 5$ which gives a value of $C = -35$. Thus $S(t) = 40 - 35\exp(-3t/20)$.

To find the time at which the salt in the tank reaches 25 lb is enough to solve for t in the equation $S(t) = 25$. This should give you $t = (20/3)\ln(7/3)$.

5. (15 points) Given the initial value problem

$$y' + 2y = 1 + t$$

with $y(0) = 0$ use Euler's method to approximate $y(1)$ with steps of length $1/3$.

Here we recognize first $y' = f(t, y) = 1 + t - 2y$. Since $h = 1/3$ and $(t_0, y_0) = (0, 0)$ then we have $t_1 = 1/3$, $t_2 = 2/3$, $t_3 = 1$. Therefore we need to compute y_1, y_2, y_3 using the formula $y_{n+1} = y_n + hf(t_n, y_n)$. The desire approximation will be $y(1) \cong y_3$. By plugging in you can check that $y_1 = 1/3$, $y_2 = 5/9$ and finally $y_3 = 20/27$.

6. (20 points) Find the general solution of

$$(2xy^2 + y) + (2x^2y + x - 1)y' = 0$$

in implicit form.

Let $M(x, y) = 2xy^2 + y$ and $N(x, y) = 2x^2y + x - 1$. Thus $M_y = 4xy + 1 = N_x$ for all values of (x, y) which means that the equation is exact.

This allows us to find the general solution by finding $f(x, y)$ such that $f_x = M$ and $f_y = N$. Recall that we were able to reduce this problem to finding $Q(x, y) = \int M\,dx$ and $g(y) = \int (N - Q_y)dy$ so that $f = Q + g$. Since $Q(x, y) = \int (2xy^2 + y)dx = x^2y^2 + xy$ and $g(y) = \int (2xy^2 + x - 1 - 2yx^2 - x)dy = \int (-1)dy = y$ the general solution is $x^2y^2 + xy - y = C$ where C is an arbitrary constant.

7. (5 points) What is the biggest interval where the solution of

$$(\sin t)y' + y = \exp\left(\frac{1}{1-t}\right)$$

with $y(\sqrt{2}) = 1$ is defined? (HINT: $\sqrt{2} \cong 1.4142$, $\pi \cong 3.1416$, $e \cong 2.7183$)

By rewritting the differential equation as

$$y' + \frac{1}{\sin t}y = \frac{e^{\frac{1}{1-t}}}{\sin t}$$

it is obvious that this is a first order linear equation. By the existence and uniqueness theorem is enough to look at the discontinuities of $e^{\frac{1}{1-t}}/(\sin t)$ and $1/(\sin t)$. They are $..., -2\pi, -\pi, 0, 1, \pi, 2\pi, ...$ thus the biggest interval is $(1, \pi)$ because this is the one that contains $\sqrt{2}$.