Midterm 2
MA 266
Instructor: Javier Zuniga NAME: ______________________________

No calculators or notes allowed. There are a total of 100 points. Read the exam carefully. Show your work for full credit.

1. (20 points) Write down the correct form of the particular solution using the method of undetermined coefficients (you don't need to find the coefficients!).

   (a) $y'' - 4y' + 4y = 2te^{2t}$; $y_{hom} = c_1 e^{2t} + c_2 te^{2t}$ and so $y_{par} = t^2(At + B)e^t$

   (b) $y'' + y = 3\sin t$; $y_{hom} = c_1 \cos t + c_2 \sin t$ and so $y_{par} = t(A\cos t + B\sin t)$

   (c) $y'' - 2y' = 4t^2$; $y_{hom} = c_1 + c_2 e^{2t}$ and so $y_{par} = t(At^2 + Bt + C)$

   (d) $y'' - 3y' - 4y = 3\cos t + 4e^{4t}$; $y_{hom} = c_1 e^{-t} + c_2 e^{4t}$ and so $y_{par} = A\cos t + B\sin t + Cte^{4t})$

2. (10 points) The spring-mass system is modeled by the second order linear equation with constant coefficients

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

   where $m, k > 0$ and $\gamma, F_0, \omega \geq 0$ are constants and with initial conditions $u(0) = u_0$, $u'(0) = v_0$. Right down on the available space on the right column the letter of the graph associated with the given values of the constants.

   The answers are b,e,d,a,c in that order.

3. (20 points) Find the solution to the initial value problem

$$y'' - y' - 6y = 6$$

   with initial conditions $y(0) = -1$ and $y'(0) = 1$.

   The characteristic polynomial is $r^2 - r - 6$ and so the homogeneous part of the solution has the form $y_{hom} = c_1 e^{3t} + c_2 e^{-2t}$. The form of the particular solution is $y_{par} = A$ and plugging this into the differential equation yields $A = -1$. Finally, since $-1 = y(0) = c_1 + c_2 - 1$ and $1 = y'(0) = 3c_1 - 2c_2$ then $c_1 = 1/5$ and $c_2 = -1/5$. Thus the solution to the IVP is $y(t) = (1/5)e^{3t} - (1/5)e^{-2t} - 1$.

   Remark.- A very common mistake here was to first solve the IVP using the initial conditions and then find the particular solution. This is wrong, you first need to know what the general solution looks like (the sum of the homogeneous and particular parts) and then plug-in for the the initial conditions.

4. (20 points) Find the general solution of the equation

$$y'' + 2y' + y = \frac{e^{-t}}{t^3}$$

The characteristic polynomial is $r^2 + 2r + 1$ and thus the homogeneous part of the solution is $y_{hom} = c_1 e^{-t} + c_2 t e^{-t}$. To find the particular solution we use variation of parameters. Take $y_1 = e^{-t}$ and $y_2 = te^{-t}$. This yields $W[y_1, y_2] = e^{-2t}$. It is clear that $g = e^{-1}/t^3$. Then $y_{par} = u_1 y_1 + u_2 y_2$ where $u_1 = -\int g y_2 / W = -\int e^{-t} t e^{-t} / (t^3 e^{-2t})\, dt = 1/t$ and $u_2 = \int g y_1 / W = \int e^{-t} e^{-t} / (t^3 e^{-2t}) = -t^{-2t}/2$. Therefore the general solution is $y(t) = c_1 e^{-t} + c_2 t e^{-t} + e^{-t}/(2t)$.

Remark.- Many of you try to use the method of undetermined coefficients here. This is wrong because $1/t^3$ is not a polynomial and therefore is not covered by the rules given in class. That is why variation of parameters was your only option here.

5. (20 points) Find the general solution of the equation

$$t^2 y'' - t y' + y = 0$$

if $y_1(t) = t$ is a solution.

We apply reduction of order. Assume $y = tv$ and thus $y' = v + tv'$ and $y'' = 2v' + tv''$. Putting this into the differential equation yields $tv'' + v' = 0$. This is a first order linear equation or a first order separable equation on v'. Since $tv'' + v' = (tv')' = 0$ then $tv' = c_1$ or $v' = c_1/t$ and so $v = c_1 \ln t + c_2$. The general solution is then $y = tv = c_1 t \ln t + c_2 t$.

6. (10 points) A bridge in the state of Washington has a weight of 10^6 lb. As the wind blows it exerts a force of $10^4 \cos(10t)$ lb. As a result the bridge vibrates similarly to a spring-mass system. Assume no friction. Write down the second order linear equation modeling this vibration. What should be the value of k for the system to enter a state of resonance and thus make the bridge collapse? (Assume $g = 32$ ft/sec^2)

Since weight is related to mass by $w = mg$ this yields $m = 10^6/32$. Thus the linear equation is

$$\frac{10^6}{32} u'' + ku = 10^4 \cos(10t)$$

To achieve resonance we need $\omega = \omega_0$ that is $10 = \sqrt{k/m} = \sqrt{k/(10^6/32)}$ and therefore $k = 10^8/32$.