Midterm 3
MA 266
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No calculators or notes allowed. There are a total of 100 points. Read the exam carefully. Show your work for full credit.

1. (20 points) Find the general solution of $y^{(5)} + y^{(3)} = 24t$

   The roots of $r^5 + r^3 = 0$ are 0 (with multiplicity 3), i, $-i$. Thus the homogeneous part of the solution is

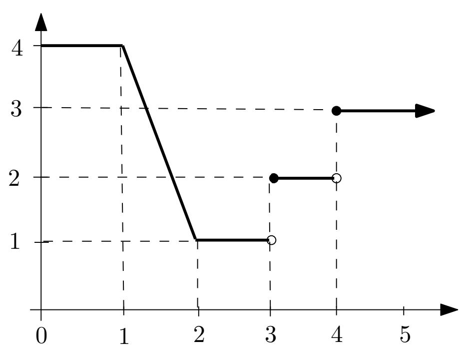
$$y_{hom}(t) = c_1 + c_2 t + c_3 t^2 + c_4 \cos t + c_5 \sin t$$

   The particular solution must have the form $y_{par}(t) = At^4 + Bt^3$. By differentiating and plugging-in back into the equation we get $A = 1$ and $B = 0$. Finally

$$y_{gen} = y_{hom} + y_{par}$$

2. (20 points) Express the function in terms of unit step functions.

   (a)



$$1 - u_1 + (u_1 - u_2)(7 - 3t) + u_2 + u_3 + u_4$$

   (b)

$$f(t) = \begin{cases} \frac{1}{2} & 0 \leq t < 1 \\ t^2 - \frac{1}{2} & 1 \leq t < 2 \\ 4 - \frac{e^{2-t}}{2} & 2 \leq t \end{cases}$$

$$\frac{1}{2}(1 - u_1) + (u_1 - u_2)(t^2 - \frac{1}{2}) + (u_2)(4 - \frac{e^{2-t}}{2})$$

3. (20 points) Consider the initial value problem given by $y'' + 3y' + 2y = f(t)$ with $y(0) = 1$ and $y'(0) = -3$. Find $f(t)$ if the solution is

$$y(t) = 2e^{-2t} - e^{-t} + u_2(t)(e^{-t} - e^{-2t})$$

The easiest way to find the answer is by direct differentiation. There are two properties you need to know: $u_c'(t) = \delta(t-c)$ and $f(t)\delta(t-c) = f(c)\delta(t-c)$.

Notice that $y'' + 3y' + 2y = 0$ has as solution

$$y_{hom} = c_1 e^{-t} + c_2 e^{-2t}$$

Thus we only need to compute the derivatives of $x(t) = u_2(t)(e^{-t} - e^{-2t})$. Using the properties stated on the previous paragraph we get

$$x'(t) = \delta(t-2)(e^{-2} - e^{-4}) + u_2(t)(-e^{-t} + 2e^{-2t})$$

and

$$x''(t) = \delta'(t-2)(e^{-2} - e^{-4}) + \delta(t-2)(2e^{-4} - e^{-2}) + u_2(t)(e^{-t} - 4e^{-2t})$$

Finally, we get

$$f(t) = x''(t) + 3x'(t) + 2x(t) = \delta'(t-2)(e^{-2} - e^{-4}) + \delta(t-2)(2e^{-4} - e^{-2})$$

4. (10 points) Find $f(t)$ if

$$f(t) + 4\int_0^t f(x)\cos(2t-2x)\,dx = 1$$

Taking the Laplace transform the equation turns into

$$F(s) + 4F(s) \cdot \frac{s}{s^2+4} = \frac{1}{s}$$

By solving for $F(s)$ and applying partial fractions we obtain

$$F(s) = \frac{1}{s} - \frac{4}{(s+2)^2}$$

Thus $f(t) = 1 - 4te^{-2t}$.

5. (10 points) Blue whales feed almost exclusively on krill and krill feeds on plankton. In the absence of the other species the number of blue whales decreases at a rate equal to twice the amount of whales alive per year, the number of krill decreases at a rate equal to three times the amount of krill per year, and the number of plankton increases at a rate equal to the amount of plankton per year. The rate of change of the whale population is increased per year by an amount equal to their interaction with krill (they eat them!) and the rate of change of the krill population is decreased by the same factor. The rate of change of the krill population is increased per year by an amount equal to twice their interaction with plankton and the rate of change of the plankton population is decreased by the same factor. Write down a system of differential equations that models the population of blue whales, krill and plankton.

If $W(t)$, $K(t)$ and $P(t)$ represent the population of whales, krill and plancton per year recall that interaction are mesure by the product of the two species interaction. So the amound of interaction between whales and krill is mesure by $W(t) \cdot K(t)$. Thus we can write down the system

$$\frac{dW}{dt} = -2W + WK$$
$$\frac{dK}{dt} = -3K - WK + 2KP$$
$$\frac{dP}{dt} = P - 2KP$$

6. (20 points) Find the solution to the initial value problem given by $x_1' = -x_1 - x_2$, $x_2' = -x_1 - x_2$ with $x_1(0) = 1$, $x_2(0) = 1$.

The system can be expressed in matrix form as

$$\vec{x}' = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \vec{x}$$

The characteristic polynomial is then $r^2 + 2r$ and thus the e-values are 0 and -2. The associated e-vectors are $(1, -1)$ and $(1, 1)$. Thus the general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

By plugging-in the initial conditions you can check that

$$x_1(t) = x_2(t) = e^{-2t}$$