Practice Midterm 1 Solutions
MA 266
Lecturer: Javier NAME: ____________________

No calculators or notes allowed. Show your work.

1. (10 points) Determine the order of the given differential equation and state whether it is linear or non-linear.

   (a) $y''' + y'' + y = 1$ order 3, linear
   (b) $y'' + \sin(t+y) = \cos t$ order 2, non-linear
   (c) $y' + ty^2 = 0$ order 1, non-linear
   (d) $(y')^{yy''} + \frac{1}{t^2} = 0$ order 2, non-linear
   (e) $t^3 y''' + \frac{y'}{\cos t} = e^{\sqrt{t}}$ order 3, linear

2. (10 points) Find the implicit solution of

$$\frac{dy}{dx} = \frac{x^2 + \cos x}{e^y + y}$$

   In this case it is easy to separate the variables so as to obtain

$$(e^y + y)dy = (x^2 + \cos x)dx$$

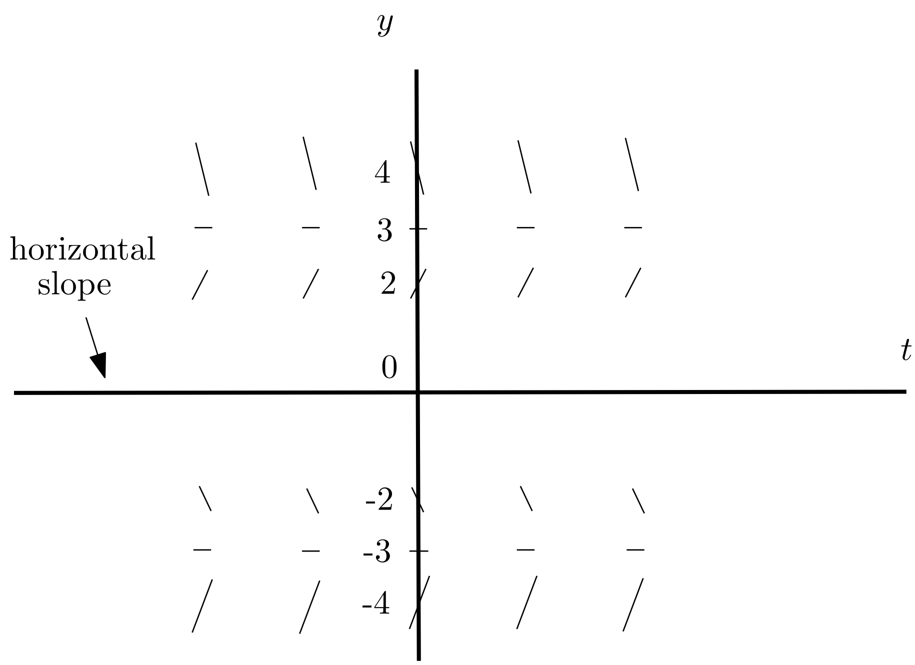
   so integrating gives the implicit solution

$$e^y + \frac{y^2}{2} = \frac{x^3}{3} + \sin x + C$$

3. (10 points) Given the equation

$$y' = y(9 - y^2)$$

   (a) Sketch the direction field
   Since this is an autonomous equation it is enough to graph the slopes on a vertical line and then copy the result along the t axis. The equation can be rewritten as $y' = y(3-y)(3+y)$ therefore it is convenient to consider the values $y = -4, -3, -2, 0, 2, 3, 4$. The corresponding slopes are $y' = 28, 0, -10, 0, 10, 0, -28$. The direction field will look like



(b) Find the equilibrium solutions and state their stability

Since $y' = y(3-y)(3+y)$ the equilibrium solutions correspond with the zeroes of the polynomial, these are $y = -3, 0, 3$ and they are stable, un-stable and stable respectively by looking at the previous direction field.

4. (25 points) Find the solution to the initial value problem

$$ty' + 3y = \frac{1}{t^2 + t^4}$$

with initial condition $y(1) = \pi/2$. Find also the biggest interval where this solution is defined.

First notice that we need to rewrite the equation as $y' + (3/t)y = 1/(t^3 + t^5)$. Now it is clear that the integrating factor is $\mu(t) = \exp(\int(3/t)dt) = t^3$. Multiplying the equation by $\mu(t)$ allow us to write $(y \cdot t^3)' = 1/(1+t^2)$. Integration yields

$$y(t) = \frac{1}{t^3}(\arctan(t) + C)$$

But $y(1) = \pi/2$ implies that $\pi/2 = \arctan(1) + C = \pi/4 + C$ and hence $C = \pi/4$. To find where this solution is defined we can just look at the solution or look at the functions $p(t)$ and $q(t)$ (because this is a linear ODE). The only discontinuity appears at $t = 0$ and so the biggest interval is $t > 0$ because it must contain $t = 1$.

5. (15 points) Given the initial value problem

$$y' = 3 + t - y$$

with $y(0) = 1$ use Euler's method to approximate $y(3)$ with steps of length one.

Since $h = 1$ and we need to approximate $y(3)$ we will need to compute y_1, y_2, y_3 with this last number being the required approximation. Since $(t_0, y_0) = (0, 1)$ and $f(t_0, y_0) = f(0, 1) = 3 + 0 - 1 = 2$ we obtain $y_1 = y_0 + f(t_0, y_0)h = 1 + (2)(1) = 3$. For the second step we need to compute $f(t_1, y_1) = f(1, 3) = 3 + 1 - 3 = 1$ and so $y_2 = 3 + (1)(1) = 4$. Finally $y_3 = y_2 + f(t_2, y_2)h = 4 + (3 + 2 - 4)(1) = 5$. This implies that $y(3) \cong y_3 = 5$.

6. (25 points) Find the general solution of

$$(y \cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0$$

By setting $M = y \cos x + 2xe^y$, $N = \sin x + x^2 e^y - 1$ it is easy to verify that the equation is exact since

$$M_y = \cos x + 2xe^y = N_x$$

Then we can find $Q(x, y) = \int M dx = y \sin x + x^2 e^y$ and thus $Q_y = \sin x + x^2 e^y$. Since $g'(y) = N - Q_y = -1$ we can take $g(y) = -y$ by integration. Thus $f = Q + g$ and the general solution is

$$y \sin x + x^2 e^y - y = C$$

7. (5 points) Initially a tank holds 50 gal of pure water. A salt solution containing 1/2 lb of salt per gal runs into the tank at the rate of 10 gal per minute. The well mixed solution runs out of the tank at the same rate. Let $S(t)$ be the amount of salt in the tank at time t. Find a differential equation satisfied by $S(t)$ (Do not solve the equation).

Recall the formula we use to model this phenomena: $\frac{dS}{dt}$ =rate in - rate out. The rate in is the product of concentration 1/2 with the flow in 10. For the rate out first notice that the tank maintains a constant amount of water because the water is flowing in at the same rate it is flowing out. So the rate out is the product of the concentration $S/50$ with the flow out 10. Thus the formula is

$$\frac{dS}{dt} = 5 - \frac{S}{5}$$