

## Practice Midterm 2

MA 266

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No calculators or notes allowed. Show your work.

1. (20 points) Write down the correct form of the particular solution using the method of undetermined coefficients (you don't need to find the coefficients!).

In all cases is important to find first the general solution to the homogeneous equation.

(a)  $y'' - y = e^t$ ;  $y_{hom} = c_1e^t + c_2e^{-t}$  and so  $y_{par} = Ate^t$

(b)  $y'' - y' = t$ ;  $y_{hom} = c_1 + c_2e^t$  and so  $y_{par} = t(At + B)$

(c)  $y'' - 2y' + 2y = 2e^t \sin t$ ;  $y_{hom} = e^t(c_1 \cos t + c_2 \sin t)$  and so  $y_{par} = te^t(A \cos t + B \sin t)$

(d)  $y'' - 2y' + y = t \cos t + e^{4t}$ ;  $y_{hom} = c_1e^t + c_2te^t$  and so  $y_{par} = (At + B) \cos t + (Ct + D) \sin t + Ee^{4t}$

2. (20 points) Consider the second order linear equation

$$at^2y'' + bty' + cy = 0$$

where  $a, b, c$  are constants and  $a \neq 0$ . Assuming that  $t > 0$  and  $(a - b)^2 > 4ac$  find  $r_1, r_2$  such that  $\{t^{r_1}, t^{r_2}\}$  forms a fundamental set of solutions.

If we assume that the solution is of the form  $y(t) = t^r$  where  $r$  is a constant then plugging into the differential equation gives  $at^2y'' + bty' + cy = ar(r - 1)t^r + brt^r + ct^r = 0$ . But since  $t > 0$  we can cancel all the  $t$  terms and the equation simplifies to  $ar^2 + (b - a)r + c = 0$ . The roots can be computed by the quadratic formula  $r_1, r_2 = (a - b \pm \sqrt{(b - a)^2 - 4ac})/2a$ . By hypothesis  $(a - b)^2 - 4ac > 0$  and so the formula gives us two different roots. To have a fundamental set of solutions all that is left is to check the Wronskian:  $W[t^{r_1}, t^{r_2}] = (t^{r_1})(r_2 t^{r_2 - 1}) - (t^{r_2})(r_1 t^{r_1 - 1}) = (r_2 - r_1)t^{r_1 + r_2 - 1}$  which is different from zero since  $r_2 \neq r_1$  and so  $r_2 - r_1 \neq 0$ .

3. (5 points) Find the general solution to the equation  $y'' + 2y' - 8y = 0$ .

The roots of  $r^2 + 2r - 8 = 0$  are 2 and -4 and so the general solution is  $y(t) = c_1e^{2t} + c_2e^{-4t}$ .

4. (15 points) Find the solution to the IVP:  $y'' - 9y = 3e^{3t}$  with initial conditions  $y(0) = 1$  and  $y'(0) = -1$ .

Since the roots of  $r^2 - 9 = 0$  are 3 and -3 the solution to the homogeneous part is  $y_{hom} = c_1e^{3t} + c_2e^{-3t}$ . The particular solution should have the form  $y_{par} = Ate^{3t}$  by inspection of the homogeneous solution. Plugging into the differential equation gives  $A = 1/2$ . Since the general solution is  $y(t) = c_1e^{3t} + c_2e^{-3t} + \frac{1}{2}te^{3t}$  this implies  $c_1 = 1/4$  and  $c_2 = 3/4$ .

5. (20 points) Consider the equation  $t^2 y'' - t(t+2)y' + (t+2)y = t^3$  with  $t > 0$ . Assuming we know that  $y_1 = t$  is a solution to the homogeneous part of the equation, use the method of reduction of order to find a second solution to the homogeneous part. Then use variation of parameters to find a particular solution to the original non-homogeneous equation.

For the homogeneous equation we need to take  $y = tv$  where  $v = v(t)$  is a function of  $t$ . Since  $y' = v't + v$  and  $y'' = v''t + 2v'$  putting this through the homogeneous differential equation yields  $v'' - v' = 0$ . This implies that  $v = c_1 + c_2 e^t$  and so  $y_{hom} = c_1 t + c_2 t e^t$ . To use variation of parameters set  $y_1 = t$ ,  $y_2 = t e^t$  and  $g(t) = t$  (remember you had to divide by  $t^2$  which is the term next to  $y''$ ). The Wronskian in this case is  $W[y_1, y_2] = t(e^t + t e^t) - t e^t(1) = t^2 e^t \neq 0$  since  $t > 0$ . Now  $u_1 = -\int y_2 g/W = -\int t^2 e^t / (t^2 e^t) dt = -\int dt = -t$  and  $u_2 = \int u_1 g/W = \int t^2 / (t^2 e^t) dt = -e^{-t}$  therefore  $y_{par} = u_1 y_1 + u_2 y_2 = (-t)(t) + (-e^{-t})(t e^t) = -t^2 - t$ .

6. (20 points) A mass weighting 16 lb stretches a spring 1/2 ft. The mass is pulled down 1 ft from the equilibrium position and then release. At the same time an external force of  $2 \cos \omega t$  is applied to the mass. Assuming no damping formulate an initial value problem modeling the system. What is the value of  $\omega$  for the system to enter a state of resonance? (Assume  $g=32$ )

Since  $mg = 16$  then  $m = 1/2$ . Also  $mg = kL$  or  $16 = k(1/2)$  and so  $k = 32$ . Since there is no damping  $\gamma = 0$  and the IVP is  $\frac{1}{2}y'' + 32y = 2 \cos(\omega t)$  with initial conditions  $y(0) = 1$ ,  $y'(0) = 0$ . Resonance is achieved when  $\omega$  is equal to the natural frequency of the system which is equal to  $\sqrt{k/m} = \sqrt{32 \cdot 2} = 8$ .