Practice Midterm 3
MA 266
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No calculators or notes allowed. Show your work. You can use the Table of Elementary Laplace Transforms on Page 319 of your textbook.

1. (20 points) Solve the initial value problem

$$y''' - y'' + y' - y = 1 + 4e^{-t}$$

with initial contitions $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$.

There are two ways to solve this problem: Using the characteristic polynomial and the method of undetermined coefficients or using the Laplace Transform. Here I ilustrate only the former (you should the try the later as an exercise! which one is faster?).

The characteristic polynomial for the homogeneous part is $p(r) = r^3 - r^2 + r - 1$. To find the roots we show two ways. First we can use the trick given in class: find roots on the divisors of the constant term of the polynomial. Since the constant term is 1 we can check for +1 or -1. But then $p(-1) = -4$ so -1 is not a root, however $p(1) = 0$ means this is a root and thus we can divide $p(r)$ by $r - 1$ to obtain $r^2 + 1$. Another way is to factor directly:

$$p(r) = r^3 - r^2 + r - 1 = (r^2)(r-1) + (1)(r-1) = (r^2+1)(r-1)$$

In any case, the homogeneous part of the solution is

$$y_{hom}(t) = c_1 \cos t + c_2 \sin t + c_3 e^t$$

By inspection the particular solution should have the form $y_{par}(t) = A + Be^{-t}$. Computing the derivatives and putting those through the equation gives $A = -1$ and $B = -1$. This means that a particular solution is

$$y_{par}(t) = -1 - e^{-t}$$

To obtain the solution to the IVP we have to plug into the general solution $y_{gen}(t) = y_{hom}(t) + y_{par}(t)$ the initial conditions. For this you will have to compute $y'_{gen}(t)$ and $y''_{gen}(t)$. The system thus obtained is $c_1 + c_3 = 3$, $c_3 - c_1 = 1$, $c_2 + c_3 = -2$. Solving this system we obtain the solution:

$$y(t) = \cos t - 4\sin t + 2e^t - 1 - e^{-t}$$

2. (10 points) Sketch the graph of the function:

   (a) $1 - u_1(t)$ step down at 1
   (b) $u_1(t) - u_2(t)$ step up at 1 and down at 2
   (c) $u_1(t) + u_2(t)$ two steps up, one at 1 and the next one at 2

(d) $u_1(t) \cdot u_2(t)$ step up at 2

(e) $u_1(t) \cdot u_1(t)$ step up at 1

3. (15 points) Consider the IVP $y'' + y = f(t)$ with initial conditions $y(0) = 0$, $y'(0) = 0$. If the solution is found to be $u_1(t)\sin(t-1) + u_2(t)(1 - \cos(t-2))$ find the forcing function $f(t)$. (Brownie points: What letter of the alphabet is the forcing function describing?)

   Taking the Laplace transform to the IVP gives $(s^2+1)Y(s) = F(s)$. To find $F(s)$ take the laplace transform of $y(t)$ to obtain

$$Y(s) = \frac{e^{-t}}{s^2+1} + e^{-2t}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) = \frac{e^{-t}}{s^2+1} + \frac{e^{-2t}}{s(s^2+1)}$$

   This yields

$$F(s) = (s^2+1)Y(s) = e^{-t} + \frac{e^{-2t}}{s}$$

   Finally

$$f(t) = \mathcal{L}^{-1}[F(s)] = \delta(t-1) + u_2(t)$$

4. (20 points) Find $f(t)$ if

   (a) $f(t) - \int_0^t (t-x)f(x)dx = 1$
   The equation can be written as $f(t) - f(t) * t = 1$. Taking the Laplace transform to this equation yields $F(s) - F(s)/s^2 = 1/s$. Solving for $F(s)$ we get

$$F(s) = \frac{(1/2)}{s-1} + \frac{(1/2)}{s+1}$$

   Now take the inverse Laplace transform to obtain

$$f(t) = (1/2)e^t + (1/2)e^{-1} = \cosh t$$

   (b) $f(t) * 1 = f(t) + 1$
   The Laplace transform of this equation gives $F(s)/s = F(s) + 1/s$. Solving for F gives $F(s) = -1/(s-1)$ and so $f(t) = -e^t$.

5. (15 points) Suppose we have three tanks or water. Tank 1 initially contains 40 gals of water with 10 oz of salt in it. Tank 2 initially contains 50 gals of water with 5 oz of salt in it. Tank 3 initially contains 60 gals of pure water. Water containing 5 oz/gal of salt flows into Tank 1 at a rate of 10 gal/min and the well-stirred mixture flows from Tank 1 to Tank 2 and Tank 3 at a rate of 6 oz/gal and 4 oz/gal respectively. The solution on Tanks 2 and 3 flows out to the ground at a rate of 6 oz/gal and 4 oz/gal respectively. Set up (but do not solve) and initial value problem describing the amount of salt on each tank.

This problem follows the same rule where the rate of change is equal to the rate in minus the rate out where rate is equal to flow times concentration. Let $x_1(t)$, $x_2(t)$ and $x_3(t)$ denote the amound of salt on the three tanks. This should give the system:

$$x_1' = (10)(5) - \frac{x_1}{40}$$
$$x_2' = 6\frac{x_1}{40} - 6\frac{x_2}{50}$$
$$x_3' = 4\frac{x_1}{40} - 4\frac{x_2}{60}$$

with initial conditions $x_1(0) = 10$, $x_2(0) = 5$ and $x_3(0) = 0$.

6. (20 points) Find the general solution of the system

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}(t)$$

The characteristic polynomial of can be obtained by finding the roots of

$$\det \begin{bmatrix} 2-r & -1 \\ 3 & -2-r \end{bmatrix} = (2-r)(-2-r) - (-1)(3) = r^2 - 1 = (r-1)(r+1)$$

The eigenvalues are then $r = 1$ and $r = -1$. To find the eigenvalues we solve the following systems:

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvectors are then

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

respectively. The general solution is then

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$